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## PREFACE

These proceedings contain a selection of the lectures given in the seminar 'Mathematical Structures in Field Theories', held at the University of Amsterdam during the last few years.

Chapter 1 by G.M. Tuynman explains the ideas of prequantization and is intended as a general introduction for the non-specialist.

In the second chapter P.G. Vroegindeweyj introduces the space-time algebra, which is the Clifford algebra associated to Minkowski space. The main goal is to give a description of the Dirac equation in terms of this space-time algebra.

Chapter 3 by C. Dullemond deals with quark confinement. The author introduces a classical model with induced metric in order to describe this phenomenon.

The fourth chapter by G.G.A. Bäuerle is devoted to the Frenkel-Kac-Segal mechanism, which provides a way to introduce gauge fields. After a review of the bosonic string in Minkowski space the author treats a special case of toroidal compactification, which gives rise to two possible affine Kac-Moody algebras as spectrum generating algebras.

The next chapter by N. van Eck gives an introduction to the theory of Hopf algebras. It may serve as first step in the study of quantum groups.

The last chapter by N.P. Landsman deals again with quantization and presents an introduction to an analytic version of deformation quantization. The central point is to study degenerate Poisson algebras and their quantization by non-simple Jordan-Lie algebras.

We thank CWI again for the technical production of these proceedings.

The editors

E.A. de Kerf

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# What is Prequantization, and what is Geometric Quantization?

G.M. Tuynman

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This paper is intended for the non-specialist (either mathematician or theoretical physicist) with a minimum knowledge of differential geometry, classical mechanics and quantum mechanics. It explains the ideas of prequantization and geometric quantization with emphasis on prequantization. In short, prequantization shows that the *mathematical* models of classical mechanics and quantum mechanics are much more alike than one should expect from the conventional formation of these theories. Geometric quantization uses these similarities between classical and quantum mechanics to extend the notion of canonical quantization to general systems in classical mechanics. To appreciate this, one should know that canonical quantization is only applicable to systems which are trivial in the mathematical sense, e.g. a phase space  $\mathbb{R}^{2n}$ .

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## 1. PHYSICAL SYSTEMS, STATES AND MODELS

Theoretical physics is the science which tries to find rules, called physical laws, according to which a certain class of natural phenomena behave. Moreover, it tries to formulate these rules in terms of mathematical models. It is therefore very important to make a distinction between the natural phenomena, the physical laws and the mathematical models.

A mathematical model consists usually of several items, e.g. sets (topological spaces, vector spaces, differentiable manifolds), special objects related to these sets (e.g. a metric tensor, a symplectic 2-form), variables denoting elements of these sets (local coordinates such as position or pressure) and certain equations in these variables (e.g. ' $PV=RT$ ' (Boyle - Gay Lussac) or ' $F=m d^2r/dt^2$ ' (Newton)). I will call these items together the *mathematical ingredients* of the model. The *interpretation* (also called the semantics) of such a model relates (some of) the mathematical ingredients to the real world, for instance to the position of the moon in the sky, or to the temperature of the water for your tea (it should be boiling!). Since all things influence each other, it follows that if we want to describe a natural phenomenon, we have to include the whole universe in our description. Clearly this is a task too complex to accomplish, so one has to make some approximations to reality. When studying a phenomenon (motion of the moon, boiling of water) one idealises the situation by neglecting those things which are supposed to be unimportant to the object of study and in fact one neglects as much as possible in order to

retain only the essential features of the phenomenon one is studying. The result of neglecting all irrelevant items is called the *physical system*, hence a physical system is an idealisation of reality. Examples of physical systems are a single free particle where one imagines a universe containing only one particle, or two particles with gravitational interaction where one imagines a universe containing only these two particles (e.g. when studying the motion of the moon around the earth these two particles represent the earth and the moon). It is such a physical system, such an idealised situation, which is described by a mathematical model. Hence in the interpretation of the mathematical model, i.e. in relating the model to reality, one has to perform two steps: first of all to relate mathematical concepts to concepts of the idealised physical system and then relating this ideal situation to reality.

When constructing a mathematical model for a physical system, one usually includes all possible states of the system under consideration. How a given state is described depends on the actual model. In some models there is a set of which the elements correspond bijectively to the possible states of the system, e.g. the symplectic model in which the points of the phase space represent the different states of the system. In some models there is a set of which the points describe more than one state of the system, e.g. the Newton model in which figures a set, called the configuration space, of which the elements denote the positions of the particles of the system, but where one also needs the velocities of the particles to distinguish between different states. Finally there exist models in which figures a set of which different elements describe the same state, e.g. quantum mechanics in which vectors of a Hilbert space which differ by a complex constant denote the same state.

## 2. CLASSICAL MECHANICS

This section is devoted to the mathematical models for classical mechanics; a review is given of the Newton formulation, Lagrange and Hamilton formulation and the symplectic formulation. For the latter three it is indicated how they can be 'derived' from the Newton formulation.

### 2.1. Newton mechanics

One can say that the origin of classical mechanics in its contemporary form is the theory of Isaac Newton which tries to describe natural phenomena in daily life (falling apples, the motion of the moon etc.) The main constituent of this theory is the well known equation of motion (Newton's second law):

$$\mathbf{F} = m\mathbf{a}$$

which tells us that the second derivative  $\mathbf{a}$  of the position of an object (where  $\mathbf{a}$  is the acceleration of the object) times the mass  $m$  of that object is equal to the applied force  $\mathbf{F}$  (a vectorial equation). If this equation is not to be a tautology, the force applied to an object should be known 'independently' of the object itself, otherwise 'force' would be a redefinition of 'acceleration'.

In the above model the mathematical ingredients are the space  $\mathbf{R}^3$ , representing all possible position of  $\mathbf{r}$  of the object in 'our 3-dimensional



space', and a function  $\mathbf{F}:\mathbb{R}^3\rightarrow\mathbb{R}^3$ , representing the force  $\mathbf{F}(\mathbf{r})$  exerted on the object when it is at the position  $\mathbf{r}$ . The model is completed by the equations of motion, which state that the various position  $\mathbf{r}(t)$  of the object at different times  $t$  satisfy the second order differential equation:

$$\mathbf{F}(\mathbf{r}(t)) = m \frac{d^2 \mathbf{r}}{dt^2}(t) \quad (2.1)$$

In this way the model can easily be extended to a system of  $N$  objects: the mathematical ingredients are the space  $\mathbb{R}^{3N}$ , a function  $\mathbf{F}:\mathbb{R}^{3N}\rightarrow\mathbb{R}^{3N}$  and the equation of motion (2.1). The interpretation is that a point  $\mathbf{r}=(\mathbf{r}_1, \dots, \mathbf{r}_N)$  represents the positions of the  $N$  objects in our 3-dimensional space, that  $\mathbf{F}(\mathbf{r})$  represents the force exerted on the objects when they are at the positions specified by  $\mathbf{r}$  and that the positions  $\mathbf{r}(t)$  at different times satisfy the equations of motion (2.1).

Newton himself derived an expression for the gravitational force in a 2-particle system, stating that the force  $\mathbf{F}_{12}$  exerted by an object 1 on an object 2 is proportional to the product of their masses divided by the square of their distance and directed towards the first object:

$$\mathbf{F}_{12}(\mathbf{r}_1, \mathbf{r}_2) = G \frac{m_1 m_2}{|\mathbf{r}_2 - \mathbf{r}_1|^3} (\mathbf{r}_1 - \mathbf{r}_2) \quad (\text{gravitational force})$$

The position  $\mathbf{r} \in \mathbb{R}^6$  then is specified as  $\mathbf{r}=(\mathbf{r}_1, \mathbf{r}_2)$  with  $\mathbf{r}_j \in \mathbb{R}^3$  the position of the  $j$ -th particle and the force  $\mathbf{F}:\mathbb{R}^6\rightarrow\mathbb{R}^6$  is given by  $\mathbf{F}(\mathbf{r}_1, \mathbf{r}_2)=(\mathbf{F}_{21}(\mathbf{r}_1, \mathbf{r}_2), \mathbf{F}_{12}(\mathbf{r}_1, \mathbf{r}_2))$  with  $\mathbf{F}_{21} = -\mathbf{F}_{12}$ ; the corresponding equations of motion (2.1) are  $m_1 d^2 \mathbf{r}_1 / dt^2 = \mathbf{F}_{21}(\mathbf{r}_1, \mathbf{r}_2)$  and  $m_2 d^2 \mathbf{r}_2 / dt^2 = \mathbf{F}_{12}(\mathbf{r}_1, \mathbf{r}_2)$ .

## 2.2. The Lagrange formulation

In the course of history the mathematical model of classical mechanics is modified to incorporate new ideas from theoretical physics. The first modification is that one supposes that the force function  $\mathbf{F}$  can be derived from a potential  $V:\mathbb{R}^{3N}\rightarrow\mathbb{R}$  by:

$$\mathbf{F}(\mathbf{r}) = -\partial V / \partial \mathbf{r}$$

This simplifies the search for a correct model for a given system enormously, because one now has to find only one function of  $3N$  variables instead of  $3N$  functions of  $3N$  variables. However, in this model the equations of motion depend manifestly on the choice of a coordinate system (for a general potential only orthogonal transformations of  $\mathbf{r}$  preserve the form of the equations of motion  $m d^2 \mathbf{r} / dt^2 = -\partial V / \partial \mathbf{r}$ ). Moreover, an initial position  $\mathbf{r}$  of the system (of  $N$  objects) does not determine a unique solution of the equations of motion because the equations of motion are *second* order differential equations in  $\mathbf{r}$ . On the other hand, if the initial position  $\mathbf{r}$  and the initial velocity  $\mathbf{v} = d\mathbf{r}/dt$  are known, then the equations of motion can be solved uniquely.

The Lagrange formulation of classical mechanics solves the above mentioned problems. Its mathematical ingredients are a space  $\mathbb{R}^{6N} = \mathbb{R}^{3N} \times \mathbb{R}^{3N}$  with coordinates  $(\mathbf{r}, \mathbf{v})$ , a function  $L:\mathbb{R}^{6N}\rightarrow\mathbb{R}$  (called the Lagrangian of the system) and

the so called Euler-Lagrange equations of motion:

$$\frac{d\mathbf{r}}{dt}(t) = \mathbf{v}(t) \quad \frac{\partial L}{\partial \mathbf{r}}(\mathbf{r}(t), \mathbf{v}(t)) = \frac{d}{dt} \left( \frac{\partial L}{\partial \mathbf{v}}(\mathbf{r}(t), \mathbf{v}(t)) \right) \quad (2.2)$$

The interpretation is that a point  $(\mathbf{r}, \mathbf{v})$  represents the positions and the velocities of the  $N$  objects (at a given time) and that the time evolution of the system (i.e. the positions and the velocities at different times) is given by the Euler - Lagrange equations of motion. A system described in the Newton formulation on  $\mathbb{R}^{3N}$  by a potential  $V$  is described in the Lagrange formulation on  $\mathbb{R}^{6N}$  by a Lagrangian  $L(\mathbf{r}, \mathbf{v}) = \frac{1}{2} m |\mathbf{v}|^2 - V(\mathbf{r})$ .

The main advantage of the Lagrange formulation is that it is 'independent' of the choice of coordinates. More specific: suppose  $\mathbf{r}'$  is another set of coordinates on  $\mathbb{R}^{3N}$ , i.e.  $\mathbf{r}' = \mathbf{r}'(\mathbf{r})$ , then we have an associated change of coordinates  $(\mathbf{r}, \mathbf{v}) \rightarrow (\mathbf{r}', \mathbf{v}')$  given by:

$$\mathbf{r}' = \mathbf{r}'(\mathbf{r}) \quad \mathbf{v}' = \mathbf{v}'(\mathbf{r}, \mathbf{v}) = \left( \frac{\partial \mathbf{r}'}{\partial \mathbf{r}} \right) \mathbf{v}. \quad (2.3)$$

Now if we define the function  $L'$  as the function  $L$  expressed in the new coordinates  $(\mathbf{r}', \mathbf{v}')$ , i.e.  $L(\mathbf{r}, \mathbf{v}) = L'(\mathbf{r}'(\mathbf{r}), \mathbf{v}'(\mathbf{r}, \mathbf{v}))$ , then solutions of the Euler - Lagrange equations (2.2) are mapped under the transformation  $(\mathbf{r}, \mathbf{v}) \rightarrow (\mathbf{r}', \mathbf{v}')$  to solutions of (2.2) in which  $(\mathbf{r}, \mathbf{v})$  and  $L$  are replaced by  $(\mathbf{r}', \mathbf{v}')$  and  $L'$ .

In modern differential geometry one recognises the transformation (2.3) as the transformation of local coordinates in the tangent bundle of a manifold (in this case the tangent bundle  $T\mathbb{R}^{3N} \cong \mathbb{R}^{6N}$  of  $\mathbb{R}^{3N}$ ). This suggests a generalisation of the mathematical ingredients of the model: instead of  $\mathbb{R}^{6N}$  one considers the tangent bundle  $TQ$  of a manifold  $Q$  and the function  $L$  becomes a function  $L: TQ \rightarrow \mathbb{R}$ . The independence of the Euler - Lagrange equations (2.2) under the coordinate transformation (2.3) guarantees that a solution of (2.2) on a local chart corresponds smoothly to a solution of (2.2) on another (intersecting) local chart.

From the point of view of physics one can ask whether this generalisation of the mathematical model has any relevance in the description of actual physical systems. The answer is affirmative if we interpret the manifold  $Q$  in the generalised model as the configuration space of the system, i.e. the space which describes the possible positions of the system. As an example one can think of a system of two masses joined by an inflexible, weightless rod of fixed length  $\rho$ ; the possible positions of this system are described by the subspace  $Q$  of  $\mathbb{R}^6$  defined by the equation  $|\mathbf{r}_1 - \mathbf{r}_2| = \rho$ , i.e.  $Q \cong S^2 \times \mathbb{R}^3$ .

### 2.3. The Hamilton formulation

The Hamilton formulation is another formulation of classical mechanics which solves the above mentioned problems of the Newton formulation with a potential function. It has the advantage that its equations of motion are manifestly first order differential equations. Its mathematical ingredients are a space  $\mathbb{R}^{6N} = \mathbb{R}^{3N} \times \mathbb{R}^{3N}$  (which is called the phase space of the system) with coordinates  $(\mathbf{r}, \mathbf{p})$ , a function  $H: \mathbb{R}^{6N} \rightarrow \mathbb{R}$  (the Hamiltonian of the system) and the

Hamilton equations of motion:

$$\frac{d\mathbf{r}}{dt}(t) = \frac{\partial H}{\partial \mathbf{p}}(\mathbf{r}(t), \mathbf{p}(t)) \quad \frac{d\mathbf{p}}{dt}(t) = -\frac{\partial H}{\partial \mathbf{r}}(\mathbf{r}(t), \mathbf{p}(t)) \quad (2.4)$$

The interpretation of this mathematical model is that a point  $(\mathbf{r}, \mathbf{p})$  represents the position  $\mathbf{r}$  and the associated momentum  $\mathbf{p}$  (usually  $\mathbf{p}$  is just  $m\mathbf{v}$ ) of the  $N$  particles, the function  $H$  represents the total energy of the system and the time evolution of the system is governed by the Hamilton equations of motion (2.4). A physical system described in the Newton formulation with  $\mathbb{R}^{3N}$  and potential function  $V$  is described in the Hamilton formulation by  $\mathbb{R}^{6N}$  and Hamiltonian function  $H(\mathbf{r}, \mathbf{p}) = |\mathbf{p}|^2/2m + V(\mathbf{r})$  (with indeed  $\mathbf{p} = m\mathbf{v} = m d\mathbf{r}/dt$ ).

The Hamilton equations of motion are first order differential equations and a well known existence theorem tells us that for all smooth functions  $H$  there exist (locally) solutions for (2.4). Moreover, the Hamilton equations are also independent of the specific choice of coordinates, i.e. if we replace the coordinates  $(\mathbf{r}, \mathbf{p})$  by  $(\mathbf{r}', \mathbf{p}')$  given by:

$$\mathbf{r}' = \mathbf{r}'(\mathbf{r}) \quad \mathbf{p}' = \left(\frac{\partial \mathbf{r}'}{\partial \mathbf{r}}\right)^{-1 \text{transp.}} \mathbf{p} \quad (2.5)$$

where  $\mathbf{r}'(\mathbf{r})$  is any coordinate transformation on  $\mathbb{R}^{3N}$ , then solutions of the Hamilton equations in the unprimed coordinates correspond to solutions of the Hamilton equations in the primed coordinates.

In (2.5) one can recognise the transformation of one set of local coordinates of a cotangent bundle  $T^*Q$  to another set of local coordinates, where  $\mathbf{r}$  and  $\mathbf{r}'$  are local coordinates on  $Q$  and where  $\mathbf{p}$  and  $\mathbf{p}'$  are the associated coordinates in the fibres. As in the Lagrange formulation one now can generalise the mathematical model to an arbitrary configuration space  $Q$  as follows: the ingredients are the cotangent bundle  $T^*Q$ , a function  $H: T^*Q \rightarrow \mathbb{R}$  and the Hamilton equations of motion (2.4). The independence of the equations (2.4) under the coordinate transformation (2.5) guarantees that the solutions of the Hamilton equations are smooth curves on  $T^*Q$ . The same arguments as for the Lagrange formulation show that this generalisation of the mathematical model is relevant for the description of real systems.

Since both the Lagrange formulation and the Hamilton formulation can easily be extended to a general configuration space  $Q$ , which yields in the Lagrange formulation the tangent space  $TQ$  and in the Hamilton formulation the cotangent space  $T^*Q$ , one might think that these two formulations are equivalent. Unfortunately this is not true for a general Lagrangian  $L$  on  $TQ$  or a general Hamiltonian  $H$  on  $T^*Q$ . Going from the Lagrange formulation to the Hamilton formulation one can show that *if* the mapping  $(\mathbf{r}, \mathbf{v}) \rightarrow (\mathbf{r}, \mathbf{p})$  defined by  $\mathbf{p}(\mathbf{r}, \mathbf{v}) = \partial L(\mathbf{r}, \mathbf{v}) / \partial \mathbf{v}$  is invertible, *then* the system described by the Lagrangian  $L$  on  $TQ$  is described in the Hamilton formulation on  $T^*Q$  by the Hamiltonian  $H$  given by  $H(\mathbf{r}, \mathbf{p}) = \mathbf{p} \cdot \mathbf{v}(\mathbf{r}, \mathbf{p}) - L(\mathbf{r}, \mathbf{v}(\mathbf{r}, \mathbf{p}))$ . In the other direction a similar situation holds: *if* the mapping  $(\mathbf{r}, \mathbf{p}) \rightarrow (\mathbf{r}, \mathbf{v})$  defined by  $\mathbf{v}(\mathbf{r}, \mathbf{p}) = \partial H(\mathbf{r}, \mathbf{p}) / \partial \mathbf{p}$  is invertible, *then* the system described by the Hamiltonian  $H$  on  $T^*Q$  is described in the Lagrange formulation on  $TQ$  by the Lagrangian

$$L(\mathbf{r}, \mathbf{v}) = \mathbf{p}(\mathbf{r}, \mathbf{v}) \cdot \mathbf{v} - H(\mathbf{r}, \mathbf{p}(\mathbf{r}, \mathbf{v})) = \mathbf{p} \cdot \partial H / \partial \mathbf{p} - H.$$

#### 2.4. A coordinate free Hamilton formulation

Once it is known that the solutions of the Hamilton equations define smooth curves on the manifold  $T^*Q$ , one would like an intrinsic formulation, independent of local coordinates. The first remark one can make is that the Hamilton equations can be interpreted as the local expression for the flow of a vector field on  $T^*Q$  and that the invariance of these equations under the coordinate transformation (2.5) just state that it is a global vector field on  $T^*Q$ . This vector field is called  $X_H$  since it obviously depends on the Hamilton function  $H$ . We thus have a global vector field  $X_H$  on  $T^*Q$  associated to the global function  $H$  on  $T^*Q$ , but its definition depends on local coordinates. To define  $X_H$  in a way independent of coordinates one introduces a 2-form  $\omega$  on  $T^*Q$ ; using local coordinates  $(r_i)$  on  $Q$  and associated coordinates  $(p_i)$  in the fibres of  $T^*Q$ ,  $\omega$  is expressed as:

$$\omega = \sum_j dp_j \wedge dr_j \quad (2.6)$$

which is indeed a global 2-form on  $T^*Q$  (and which can be defined in an intrinsic way). Using this 2-form  $\omega$  the vector field  $X_H$  is defined by the intrinsic equation:

$$i(X_H)\omega + dH = 0 \quad (2.7)$$

where  $i(Y)$  is the substitution operator of vector fields in forms. In this way the ingredients of the model become the space  $T^*Q$  with the 2-form  $\omega$ , the (smooth) function  $H$  and the equation (2.7). The interpretation of the model is that a point  $(\mathbf{r}, \mathbf{p})$  in  $T^*Q$  represents the system, i.e.  $\mathbf{r}$  denotes the position of the system and  $\mathbf{p}$  denotes the associated momentum, and the time evolution is given by the flow  $\phi_\tau$  of the vector field  $X_H$  defined by (2.7).

For those who are unfamiliar with modern differential geometry, we give a translation in more down to earth notations in case of the phase space  $\mathbb{R}^{6N}$ . A vector field  $X$  on  $\mathbb{R}^{6N}$  'is' a mapping  $X: \mathbb{R}^{6N} \rightarrow \mathbb{R}^{6N}$  which assigns to a point of  $\mathbb{R}^{6N}$  a vector (indicating a direction of motion). With the same abuse of notation, the 1-form  $dH$  'is' the vector of all partial derivatives:  $dH = (\partial H / \partial \mathbf{r}, \partial H / \partial \mathbf{p}) \in \mathbb{R}^{6N}$ , and the 2-form  $\omega$  'is' a  $6N \times 6N$  matrix:

$$\omega = \begin{bmatrix} 0 & I_{3N} \\ -I_{3N} & 0 \end{bmatrix} \quad \text{where } I_{3N} \text{ denotes the } 3N \times 3N \text{ identity matrix.}$$

With these notations, the equation (2.7) becomes the vector equation  $\omega X_H + dH = 0$ , which has a unique solution for  $X_H$  since  $\omega$  is invertible:

$$X_H = (\partial H / \partial \mathbf{p}, -\partial H / \partial \mathbf{r}).$$

The flow  $\phi_\tau$  of the vector field  $X_H$  is a mapping  $\phi_\tau: \mathbb{R}^{6N} \rightarrow \mathbb{R}^{6N}$  satisfying

$$\frac{\partial \phi}{\partial \tau} \Big|_{\tau=0}(\mathbf{r}, \mathbf{p}) = X_H(\mathbf{r}, \mathbf{p}) \quad \text{and} \quad \phi_0(\mathbf{r}, \mathbf{p}) = (\mathbf{r}, \mathbf{p}).$$

If we now write  $\phi_t(\mathbf{r}, \mathbf{p}) = (\mathbf{r}(t), \mathbf{p}(t))$ , then  $(\mathbf{r}(t), \mathbf{p}(t))$  is nothing less than a

solution of Hamilton's equations of motion with  $(\mathbf{r}, \mathbf{p})$  as initial condition, i.e. the flow  $\phi_r$  denotes the time evolution of the system.

### 2.5. The symplectic formulation and the Poisson algebra

We now want to extract the essential mathematical features of the above model. Obvious ingredients are a manifold  $M (= T^*Q)$ , a 2-form  $\omega$  on  $M$  and a function  $H$  on  $M$ . However, not all such triples  $(M, \omega, H)$  are meaningful. In the first place the 2-form  $\omega$  has to be *non-degenerate* to ensure that the equation (2.7) has a unique solution for the vector field  $X_H$ . In the second place the Poisson algebra (to be defined below) is a useful tool in theoretical physics, so one likes to have it in the generalised case too.

In the Hamilton formulation of classical mechanics on  $\mathbb{R}^{6N}$ , the Poisson algebra is the collection of all smooth functions on  $\mathbb{R}^{6N}$  (which is a vector space under pointwise addition of functions) equipped with the so called Poisson bracket of functions  $\{f, g\}$  defined by:

$$\{f, g\} = \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial g}{\partial \mathbf{r}} - \frac{\partial f}{\partial \mathbf{r}} \cdot \frac{\partial g}{\partial \mathbf{p}}. \quad (2.8)$$

With these definitions the Poisson bracket is bilinear, antisymmetric and satisfies the Jacobi identity:

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$$

hence the set of all (smooth) functions on  $\mathbb{R}^{6N}$  equipped with this bracket is an (infinite dimensional) Lie algebra which is called the Poisson algebra.

In the formulation in terms of  $T^*Q$  and  $\omega$  the Poisson algebra can be defined as the set of smooth functions on  $T^*Q$  equipped with the Poisson bracket defined by:

$$\{f, g\} = \omega(X_f, X_g) \quad (2.9)$$

where  $X_f$  (and likewise  $X_g$ ) are defined by:

$$i(X_f)\omega + df = 0 \quad (2.10)$$

which is exactly equation (2.7) with the Hamiltonian  $H$  replaced by the arbitrary function  $f$ . One can verify that this bracket indeed satisfies the Jacobi identity and reduces to (2.8) in the case  $Q = \mathbb{R}^{3N}$  (in our down to earth vector and matrix notations the righthand side of (2.9) is the inner product of the vector  $X_g$  with  $\omega X_f$ ).

In the case of a triple  $(M, \omega, H)$  with  $\omega$  a non-degenerate 2-form (to ensure that one can define vector fields  $X_f$  by means of (2.10)), the bracket (2.9) on the set of smooth functions on  $M$  does not in general satisfy the Jacobi identity. One can show that the bracket (2.9) *does* satisfy the Jacobi identity if and only if  $\omega$  is *closed*. A manifold  $M$  equipped with a closed non-degenerate 2-form  $\omega$  is called a *symplectic manifold*,  $\omega$  its *symplectic form* and the geometry of a symplectic manifold is called *symplectic geometry*.

With these preparations we can generalise the Hamilton formulation to the symplectic formulation. The ingredients are a symplectic manifold  $(M, \omega)$  and a

real valued function  $H$  on  $M$ . The interpretation is that the phase space  $M$  represents (uniquely) all different states of the system and that the time evolution is given by the flow of the vector field  $X_H$  defined by (2.7) or (2.10).

For physicists the important question is whether these mathematical exercises do have any relevance for physics. Again the answer is yes: there exist systems in physics which *can not* be described by a cotangent bundle, but which *can* be described by a triple  $(M, \omega, H)$ . A simple example of such a system (described extensively in [So1]) is the phase space of classical spin which is described by the space  $M=S^2$  (the unit sphere) and the 2-form  $\omega=\lambda \sin \theta d\theta \wedge d\phi$  (with  $\lambda$  a real parameter); the sphere  $S^2$  is clearly not a cotangent bundle because  $S^2$  is compact and a cotangent bundle is necessarily non-compact.

### 2.6. Summary of the symplectic formulation

The symplectic formulation of classical mechanics is given by the following items.

#### *Mathematical ingredients*

A manifold  $M$ , a non-degenerate closed 2-form  $\omega$  on  $M$  and a function  $H:M \rightarrow \mathbb{R}$ .

#### *Mathematical manipulations*

To each function  $f$  on  $M$  is associated a vector field  $X_f$  defined by  $i(X_f)\omega + df = 0$  and to two functions  $f, g$  on  $M$  is associated a function  $\{f, g\}$  defined by  $\{f, g\} = \omega(X_f, X_g)$ .

#### *Nomenclature*

$(M, \omega)$  is called a *symplectic manifold*,  $\omega$  its *symplectic form*; in a physical context  $M$  is called the *phase space* of the system under consideration.  $H$  is the *Hamiltonian* (total energy) of the system,  $X_f$  is the *Hamiltonian vector field* associated to the function  $f$  and  $\{f, g\}$  is the *Poisson bracket* of  $f$  and  $g$ .

#### *Interpretation*

The different possible states of the physical system under consideration are in 1-1 correspondence with the points of  $M$ . If at a given time  $t$  the system is represented by the point  $m \in M$ , then at time  $t'$  the system is represented by the point  $m' = \phi_{t'-t}(m)$  where  $\phi_\tau$  denotes the flow of the vector field  $X_H$  on  $M$ .

#### *Usefulness of the Poisson bracket*

A function  $f$  on  $M$  is called a *conserved quantity* if the function  $f \circ \phi_\tau$  is independent of the parameter  $\tau$ , where as before  $\phi_\tau$  denotes the flow of the vector field  $X_H$ ; physically speaking  $f$  is a conserved quantity if the value of the function  $f$  evaluated at a point which represent the system at a given time  $t$  does not depend on  $t$ , i.e. if  $f$  is constant in time. An elementary calculation shows that  $f$  is a conserved quantity if and only if the Poisson bracket  $\{f, H\}$  is the zero function.

*Remark 1*

The 2-form  $\omega$  must be non-degenerate to guarantee a unique solution for the defining equation of  $X_f$ ;  $\omega$  must be closed to insure that the Poisson bracket  $\{, \}$  satisfies the Jacobi identity.

*Remark 2*

At each point  $m \in M$ , after the choice of a local coordinate system around  $m$  the non-degenerate 2-form  $\omega$  can be represented by an anti-symmetric non-degenerate matrix  $\Omega$ , hence the dimension of  $M$  is even,  $\dim M = 2n$ . A famous theorem of Darboux tells us that for each point  $m \in M$  there exist local coordinates  $(r_1, \dots, r_n, p_1, \dots, p_n)$  in a neighbourhood of  $m$  such that  $\omega$  is given by  $\omega = \sum_j dp_j \wedge dr_j$ .

*Remark 3*

In our treatment (which is the usual one), the symplectic phase space plays the role of the set of initial values for the Hamilton equations. However, there exists another interpretation of the symplectic manifold  $M$  (see [So1]) in which  $M$  is interpreted as the space of movements, i.e. each point of  $M$  represents the whole movement of the system in time (c.f. the Heisenberg picture of quantum mechanics). In most cases these two interpretations coincide, but not always. An example in which they do not coincide is the Kepler problem, in which the phase space is  $(\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3$ , i.e. the cotangent space of  $\mathbb{R}^3$  without its origin, and in which the space of movements with negative energy can be identified (after a regularisation) with  $T_o S^3$ , i.e. the (co)tangent space of  $S^3$  without the zero section ([So2]).

*Example*

We can realise the Newton formulation with a potential function  $V$  on  $\mathbb{R}^{3N}$  in the symplectic formulation as follows. The phase space  $M = \mathbb{R}^{6N} \ni (\mathbf{r}, \mathbf{p})$ , the symplectic form  $\omega$  is the global form  $\omega = \sum_j dp_j \wedge dr_j$  and the Hamiltonian  $H$  is the function on  $M$  given by  $H(\mathbf{r}, \mathbf{p}) = |\mathbf{p}|^2/2m + V(\mathbf{r})$ . The Hamiltonian vector field  $X_f$  of the function  $f$  is given by:

$$X_f = \frac{\partial f}{\partial p_j} \frac{\partial}{\partial r_j} - \frac{\partial f}{\partial r_j} \frac{\partial}{\partial p_j} \cong (\partial f / \partial \mathbf{p}, -\partial f / \partial \mathbf{r}) \Rightarrow X_H(\mathbf{r}, \mathbf{p}) = (\mathbf{p}/m, -\partial V / \partial \mathbf{r})$$

whence the flow  $\phi_t$  of  $X_H$  is defined by the equations:

$$d\mathbf{r}/dt = \mathbf{p}/m \quad \& \quad d\mathbf{p}/dt = -\partial V / \partial \mathbf{r}.$$

Substitution of the first equation in the second gives us Newton's law:

$$m d^2 \mathbf{r} / dt^2 = -\partial V / \partial \mathbf{r}.$$

### 3. QUANTUM MECHANICS

In this section we present the usual mathematical model of quantum mechanics. It is shown that the mathematical model for the dynamics in quantum mechanics can be modified in such a way that the modified model fits the

description of classical mechanics in the symplectic formulation (2.6) with the one exception that the symplectic manifold is infinite dimensional. For the sake of simplicity we ignore the fact that self adjoint operators are usually not defined everywhere; in view of Stone's theorem this is not a serious omission because we are 'only' interested in the global 1-parameter group generated by such a self adjoint operator.

### 3.1. The usual formulation

The *mathematical* model of quantum mechanics is quite simple. The ingredients are a complex Hilbert space  $\mathfrak{H}$ , a self-adjoint operator  $\mathbb{H}$  on  $\mathfrak{H}$  called the Hamiltonian of the system and the Schrödinger equation on  $\mathfrak{H}$ :

$$-i\hbar\frac{\partial\psi}{\partial t}(t) = \mathbb{H}\psi(t) \quad (3.1)$$

where  $\hbar$  denotes Planck's constant  $h$  divided by  $2\pi$ .

The *interpretation* of this model is more complicated and consists of two parts: a dynamical part and a probability part.

#### *Interpretation of dynamics*

Each non-zero vector  $\psi \in \mathfrak{H}$  represents a state of the physical system under consideration, but two vectors  $\psi$  and  $\psi'$  which differ by a non-zero complex constant  $\lambda$  (i.e.  $\psi' = \lambda\psi$ ) represent the same state (and vice versa if  $\psi$  and  $\psi'$  represent the same state then they differ by a non-zero complex constant). If at time  $t_0$  the system is represented by the vector  $\psi_0$  and if  $\psi(t)$  is a solution of the Schrödinger equation (3.1) with the initial condition  $\psi(t_0) = \psi_0$  then at time  $t$  the system is represented by  $\psi(t)$ .

#### *The probability interpretation*

An observable  $f$ , i.e. a measurable quantity, is represented by a self-adjoint operator  $\underline{f}$  on  $\mathfrak{H}$ . The result of a measurement is an eigenvalue of the operator  $\underline{f}$ , but which one is unpredictable. However, a probability distribution is assigned to the possible results by means of the spectral theorem for self-adjoint operators. If (at time  $t$ ) the system is represented by the vector  $\psi$ , then the expectation value  $E(\underline{f}, \psi)$  of the result of measuring  $f$  when the system is described by  $\psi$  is given by:

$$E(\underline{f}, \psi) = \frac{\langle \psi, \underline{f}\psi \rangle}{\langle \psi, \psi \rangle} \quad (3.2)$$

where  $\langle \psi, \chi \rangle$  denotes the inner product of the vectors  $\psi, \chi \in \mathfrak{H}$ . Since  $\underline{f}$  is a self adjoint operator the expectation value  $E(\underline{f}, \psi)$  is real and moreover, it is independent of the choice of the vector  $\psi$  representing the state of the system:  $E(\underline{f}, \psi) = E(\underline{f}, \lambda\psi)$  for  $\lambda \neq 0$ .



In this paper we are only interested in the dynamic part so in the sequel we will ignore the probability interpretation, although it is the essential part of quantum mechanics: it is the main distinction between classical mechanics and quantum mechanics and, as we will see, the only one. In order to be sure that the interpretation is not contradictory with the mathematical model we have to make an important remark. The operator  $\mathbf{H}$  is self-adjoint, hence (by Stone's theorem) there exists a 1-parameter group  $U(\tau)$  of unitary transformations of  $\mathcal{H}$  of which  $i\mathbf{H}/\hbar$  is the infinitesimal generator:

$$U(\tau) = \exp(i\tau\mathbf{H}/\hbar). \quad (3.3)$$

It follows that the solutions of the Schrödinger equation (3.1) are given by  $\psi(\tau) = U(\tau)\psi$  where  $\psi$  is an arbitrary initial condition. We deduce that if at time  $t_o$  the system is represented by  $\psi_o$  then at time  $t$  it is represented by the vector  $\psi(t)$  given by:

$$\psi(t) = U(t-t_o)\psi_o = \exp(i(t-t_o)\mathbf{H}/\hbar)\psi(t_o) = \exp(it\mathbf{H}/\hbar)\psi(0). \quad (3.4)$$

Since  $U(\tau)$  is a *complex linear* transformation of  $\mathcal{H}$ , it follows that the time evolution does not contradict the ambiguity in the vector  $\psi$  representing the state of the physical system, i.e. if  $\psi$  and  $\psi'$  represent the same state at time  $t_o$ , then  $U(t-t_o)\psi$  and  $U(t-t_o)\psi'$  represent the same state, which is the state at time  $t$ .

### 3.2. An unusual formulation of quantum dynamics

If  $\psi \in \mathcal{H}$  then, in the above interpretation, all non-zero points of the complex line  $\mathbb{C}\psi = \{\lambda\psi | \lambda \in \mathbb{C}\}$  represent the same state; in fact each state of the system is represented *uniquely* by a complex line in  $\mathcal{H}$ . In mathematics the set of all (complex) lines in a (complex) vector space is called the projective space associated to the vector space, so we have found that in quantum mechanics the states of a physical system are represented uniquely by the elements of the projective Hilbert space  $\mathbb{P}\mathcal{H}$ .

We now mention without proof some facts concerning  $\mathbb{P}\mathcal{H}$  which are not generally known (the proofs can be found in [Tul]). First of all, each projective complex Hilbert space  $\mathbb{P}\mathcal{H}$  is an (infinite dimensional) complex manifold which has a canonically defined symplectic structure  $\omega$ . Secondly, since the Schrödinger equation is compatible with the ambiguity of the vector representing a state of the system, it follows that there exists a 1-parameter group  $\phi_\tau$  of diffeomorphisms on  $\mathbb{P}\mathcal{H}$  such that:

$$\pi \circ U(\tau) = \phi_\tau \circ \pi \quad \text{with } U(\tau) = \exp(i\tau\mathbf{H}/\hbar) \quad (3.5)$$

where  $\pi$  denotes the projection  $\pi: \mathcal{H} \setminus \{0\} \rightarrow \mathbb{P}\mathcal{H}$  which associates to a vector  $\psi$  the unique complex line  $\mathbb{C}\psi$  containing  $\psi$ ; in fact equation (3.5) can be regarded as the definition of the flow  $\phi_\tau$ . To complete the preliminaries we introduce the function  $E(\mathbf{H}): \mathbb{P}\mathcal{H} \rightarrow \mathbb{R}$  by:

$$E(\mathbf{H})(\pi\psi) = E(\mathbf{H}, \psi) = \frac{\langle \psi, \mathbf{H}\psi \rangle}{\langle \psi, \psi \rangle}.$$

It now turns out that the flow  $\phi_t$  is the flow of the Hamiltonian vector field  $X_{E(\mathbf{H})}$  associated to the real function  $E(\mathbf{H})$  by the symplectic form  $\omega$  according to formula (2.10). Finally, since  $\mathbf{H}$  governs the time evolution of the states in quantum mechanics, it follows from (3.4) and (3.5) that if the system is described at time  $t=0$  by  $\pi\psi_0 \in \mathbb{P}\mathcal{H}$  then it is described at time  $t$  by the point  $\phi_t(\pi\psi_0) \in \mathbb{P}\mathcal{H}$ .

We can summarise these observations by saying that the dynamics of quantum theory is described by the triple  $(\mathbb{P}\mathcal{H}, \omega, E(\mathbf{H}))$  according to the symplectic formulation of classical mechanics (if we allow infinite dimensional manifolds): the states of the system are in 1-1 correspondence with the points of  $\mathbb{P}\mathcal{H}$  and the time evolution of a state is described by the flow of the Hamiltonian vector field associated to  $E(\mathbf{H})$ . This shows that the mathematical models of classical mechanics and the dynamical part of quantum mechanics are identical and that their interpretations are also the same. The differences between classical and quantum mechanics lies in the choice of the symplectic manifolds: in classical mechanics one can use 'all' finite dimensional symplectic manifolds but in quantum mechanics one uses only (infinite dimensional complex projective spaces). We stress that the symplectic model of quantum mechanics describes only the dynamical part; for the probability part it is essential to know that the quantum mechanical 'phase space'  $\mathbb{P}\mathcal{H}$  is derived from a Hilbert space  $\mathcal{H}$ .

### 3.3. The unit sphere and phase factors

In the previous section we have seen that in quantum mechanics the elements of the Hilbert space  $\mathcal{H}$  do not represent the states of the physical system in a unique way but that the elements of the projective Hilbert space  $\mathbb{P}\mathcal{H}$  do. One prefers however to work with  $\mathcal{H}$ , because working with  $\mathbb{P}\mathcal{H}$  is rather difficult. An intermediate level between  $\mathcal{H}$  and  $\mathbb{P}\mathcal{H}$  is the unit sphere  $S\mathcal{H}$  in  $\mathcal{H}$ :

$$S\mathcal{H} = \{\psi \in \mathcal{H} \mid \langle \psi, \psi \rangle = 1\}$$

Each state of the physical system is represented by a non-zero vector  $\psi \in \mathcal{H}$ , but then it is also represented by the vector  $\psi/\|\psi\|$  (i.e.  $\psi$  divided by its norm) which is an element of the unit sphere. It follows that each state can be represented by an element of  $S\mathcal{H}$ , but there remains an ambiguity: if  $\psi \in S\mathcal{H}$  then  $e^{i\theta}\psi \in S\mathcal{H}$  and  $\psi$  and  $e^{i\theta}\psi$  represent the same state, hence a state is represented by a circle on the unit sphere  $S\mathcal{H}$ . If we restrict the projection  $\mathcal{H} \setminus \{0\} \rightarrow \mathbb{P}\mathcal{H}$  to the unit sphere we obtain a projection  $\pi: S\mathcal{H} \rightarrow \mathbb{P}\mathcal{H}$  of which the inverse images are just the circles representing the states. Two different points  $\psi, \chi$  on such a circle are related to each other by:

$$\chi = e^{i\theta}\psi$$

where  $e^{i\theta}$  is a complex number of modulus 1 usually called a phase factor. The aim is now to translate the mathematical model of quantum dynamics in the Hilbert space formulation to the unit sphere. The ultimate goal of this exercise is to show that the analogy between classical mechanics and quantum mechanics as given in the previous section (3.2) can be extended and that one can describe within the framework of classical mechanics certain phenomena which

are usually thought of as being purely quantum phenomena, i.e. phenomena which can be described only within the framework of quantum mechanics.

To translate the dynamics from  $\mathcal{H}$  to  $S\mathcal{H}$  we first observe that the Schrödinger equation can be seen as the flow of a vector field. If  $\mathbf{H}$  is the Hamiltonian of the system then we define a vector field  $V_{\mathbf{H}}$  on  $\mathcal{H}$ , i.e. a map  $\mathcal{H} \rightarrow \mathcal{H}$ , by:

$$V_{\mathbf{H}}(\psi) = (i/\hbar)\mathbf{H}\psi \quad (3.6)$$

and then the Schrödinger equation is equivalent to the equation of the flow of  $V_{\mathbf{H}}$  which is defined as  $d\psi/dt = V_{\mathbf{H}}(\psi)$ . Since  $\mathbf{H}$  is a self-adjoint operator, it follows that the vector field  $V_{\mathbf{H}}$  is tangent to the unit sphere  $S\mathcal{H}$ . Another way to see that  $V_{\mathbf{H}}$  is tangent to  $S\mathcal{H}$  is to observe that the flow of  $V_{\mathbf{H}}$  is the unitary 1-parameter group  $U(\tau) = \exp(i\tau\mathbf{H}/\hbar)$  which means in particular that the norm of a vector is conserved, i.e.  $U(\tau)$  can be seen as a 1-parameter group of diffeomorphisms of  $S\mathcal{H}$ .

The above observations show that we can represent the dynamics of quantum mechanics in a mathematical model consisting of the unit sphere  $S\mathcal{H}$  and the vector field  $V_{\mathbf{H}}$ , with the interpretation that each state of the system is represented by a circle on  $S\mathcal{H}$  (freedom of phase vectors) and that the time evolution is given by the flow of the vector field  $V_{\mathbf{H}}$  (which is indeed a vector field on  $S\mathcal{H}$ ).

#### 3.4. Summary

For ordinary quantum mechanics we have given three mathematical models for the dynamics of a quantum system. We will now summarise the essential features of these models.

##### *Ingredients of the Hilbert space formulation*

A Hilbert space  $\mathcal{H}$ , a self adjoint operator  $\mathbf{H}$  on  $\mathcal{H}$  and the Schrödinger equation.

##### *Interpretation of the Hilbert space formulation*

Each state of the system is represented by a complex line in  $\mathcal{H}$  and the time evolution of the system is given by the solutions of the Schrödinger equation (3.1) associated to the self adjoint operator  $\mathbf{H}$  (N.B. the time evolution maps complex lines into complex lines so the time evolution of the states is well defined).

##### *Ingredients of the unit sphere formulation*

The unit sphere  $S\mathcal{H}$  in  $\mathcal{H}$  and a vector field  $V_{\mathbf{H}}$  on  $S\mathcal{H}$  (associated to  $\mathbf{H}$ ).

##### *Interpretation of the unit sphere formulation*

Each state of the system is represented by a circle on  $S\mathcal{H}$  (generated by multiplication by phase factors  $e^{i\theta}$ ) and the time evolution of the system is given by the flow of the vector field  $V_{\mathbf{H}}$  (which indeed maps the circles representing states into circles representing states).

*Ingredients of the projective Hilbert space formulation*

The projective Hilbert space  $\mathbb{P}\mathcal{H}$ , its canonical symplectic form  $\omega$  and the real function  $E(\mathbb{H}): \mathbb{P}\mathcal{H} \rightarrow \mathbb{R}$ .

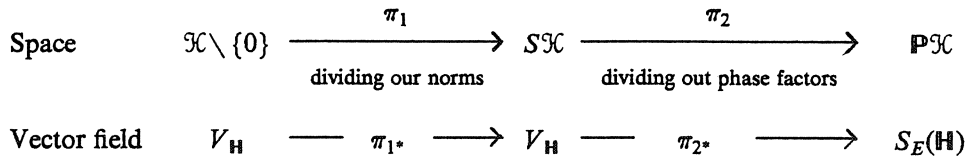
*Interpretation of the projective Hilbert space formulation*

There is a 1-1 correspondence between the states of the system and the points of the projective Hilbert space  $\mathbb{P}\mathcal{H}$ ; the time evolution of the system is given by the flow of the Hamiltonian vector field  $X_{E(\mathbb{H})}$  on  $\mathbb{P}\mathcal{H}$  defined by the symplectic form  $\omega$  and the function  $E(\mathbb{H})$  according to formula (2.10).

*Remark*

In the Hilbert space formulation of quantum mechanics the arbitrariness in the vector representing a state of the system is a non-zero complex number; in the unit sphere formulation this arbitrariness is reduced to a phase factor and finally in the projective Hilbert space formulation there is no arbitrariness at all. In the Hilbert space formulation the time evolution of the system is governed by the Schrödinger equation, or (a completely equivalent description) by the flow of the vector field  $V_{\mathbb{H}}$  on  $\mathcal{H}$ . When we descend to the unit sphere by projecting a non-zero vector  $\psi$  to the unit vector  $\psi/\|\psi\|$  we can project the vector field  $V_{\mathbb{H}}$  to a vector field on  $S\mathcal{H}$ . Since the projected vector field equals the restriction of  $V_{\mathbb{H}}$  to the unit sphere, this vector field will also be called  $V_{\mathbb{H}}$ . For the projective Hilbert space formulation we project a second time, now from  $S\mathcal{H}$  to  $\mathbb{P}\mathcal{H}$ , i.e. we identify points on  $S\mathcal{H}$  which differ by a phase factor. The vector field  $V_{\mathbb{H}}$  (now seen as a vector field on  $S\mathcal{H}$ ) can be projected to  $\mathbb{P}\mathcal{H}$  and the result is the Hamiltonian vector field  $X_{E(\mathbb{H})}$  on  $\mathbb{P}\mathcal{H}$  associated to the function  $E(\mathbb{H})$  which represents the expectation value of the Hamiltonian. Since  $X_{E(\mathbb{H})}$  is the projection of the vector field  $V_{\mathbb{H}}$  on  $\mathcal{H}$  it follows that the flow of  $X_{E(\mathbb{H})}$  is the projection of the unitary group  $U(\tau) = \exp(i\tau\mathbb{H}/\hbar)$  which is the flow of  $V_{\mathbb{H}}$  on  $\mathcal{H}$ . This shows why the Hilbert space formulation is so much easier to use than the projective Hilbert space formulation: computing  $\exp(i\tau\mathbb{H}/\hbar)$  is much easier than computing the flow of a vector field on an infinite dimensional manifold. On the other hand, the projective Hilbert space formulation is better suited for abstract considerations because there is no ambiguity between states of the system and the points which represent these states.

*Diagram*



Its flow	$\exp(i\tau\mathbf{H}/\hbar)$	$\exp(i\tau\mathbf{H}/\hbar)$ restricted to $S\mathcal{H}$	$\pi_2 \exp(\tau\mathbf{H}/\hbar)\tau_2 - 1$
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#### 4. PREQUANTIZATION

In the first part of this section the idea of prequantization of a symplectic manifold is explained using the example of the symplectic manifold  $\mathbb{R}^{2n}$ , which is the phase space of a particle in  $n$  dimensions (or, if  $n = 3N$ , the phase space of  $N$  particles in our 3-dimensional space). Although this example does not show all essential features of prequantization, it suffices to show the idea; afterwards the problems arising in the general case will be pointed out. The second part of this section will be devoted to the statement that the analogy between classical mechanics and quantum mechanics is more than the simple observation that both can be described by the symplectic formalism.

##### 4.1. Phase factors in classical mechanics

In §3 we have seen that in quantum mechanics the states of a physical system are uniquely represented by points of a projective Hilbert space  $\mathbb{P}\mathcal{H}$ , but that it is much easier to work with the Hilbert space  $\mathcal{H}$ . Since self adjoint operators generate *unitary* groups according to formula (3.3), we can concentrate without loss of generality on the unit sphere  $S\mathcal{H}$ , in which states are represented by circles generated by multiplication with phase factors. To summarise: in quantum mechanics states are represented by points of  $\mathbb{P}\mathcal{H}$  and we have a projection  $\pi: S\mathcal{H} \rightarrow \mathbb{P}\mathcal{H}$  in which the inverse image of a point is a circle  $U(1)$  of phase factors.

Prequantization duplicates this situation in classical mechanics. This was first done by L. van Hove [vH] who investigated the canonical quantization program of Dirac; later on the same ideas were developed independently by Kostant [Ko] and Souriau [So1]. Let us start with the example of the phase space  $M = \mathbb{R}^{2n}$  with coordinates  $(\mathbf{r}, \mathbf{p})$  and its canonical symplectic form  $\omega = \sum_j dp_j \wedge dr_j$ , then we can define a space  $Y$  and a projection  $\pi: Y \rightarrow M$  by:

$$Y = \mathbb{R}^{2n} \times U(1) \quad \text{and } \pi: Y \rightarrow M, (\mathbf{r}, \mathbf{p}, e^{i\theta}) \rightarrow (\mathbf{r}, \mathbf{p})$$

In this situation the inverse image of a point in  $M$  (representing a state of the physical system) is a circle of phase factors.

As far as phase factors are concerned we now have trivially duplicated the situation of quantum mechanics, but what purpose does it have? To facilitate the notation we will use the angle  $\theta$  as a coordinate on  $U(1)$  (so  $Y$  is parametrised by  $(\mathbf{r}, \mathbf{p}, \theta)$ ) and we will denote a vector field  $X$  on  $M$  as a mapping  $X: M \rightarrow \mathbb{R}^{2n}$  and a vector field  $V$  on  $Y$  as a mapping  $V: Y \rightarrow \mathbb{R}^{2n+1}$ . Now if  $f$  is any real valued function on  $M$  (e.g. the Hamiltonian  $H$ ), then its Hamiltonian vector field  $X_f$  is defined by equation (2.10) which gives us:

$$X_f(\mathbf{r}, \mathbf{p}) = \left( \frac{\partial f}{\partial \mathbf{p}}(\mathbf{r}, \mathbf{p}), -\frac{\partial f}{\partial \mathbf{r}}(\mathbf{r}, \mathbf{p}) \right). \quad (4.1)$$

We now mention without proof that the mapping  $f \rightarrow X_f$  from the Poisson

algebra to vector fields is a mapping of Lie algebras (where we use for functions the Poisson bracket and for vector fields the commutator of vector fields). One easily verifies that the kernel of this map consists of the constant functions on  $M$ , so it is not an injective map. We now can use the space  $Y$  to construct a map  $f \rightarrow V_f$  from the Poisson algebra to vector fields on  $Y$  which is an injective Lie algebra morphism and which projects on  $X_f$ . The fact that  $Y$  can be used in this way can be seen as a mathematical motivation for its introduction. Within a certain framework which will be specified later, the map  $f \rightarrow V_f$  is unique and given by:

$$V_f(\mathbf{r}, \mathbf{p}, \theta) = \left( \frac{\partial f}{\partial \mathbf{p}}(\mathbf{r}, \mathbf{p}), -\frac{\partial f}{\partial \mathbf{r}}(\mathbf{r}, \mathbf{p}), f(\mathbf{r}, \mathbf{p}) - \mathbf{p} \cdot \frac{\partial f}{\partial \mathbf{p}}(\mathbf{r}, \mathbf{p}) \right). \quad (4.2)$$

From (4.2) one deduces immediately that it is injective: constant functions are mapped to vector fields in the direction of the phase factors, and projects to  $X_f$ : omitting the  $\theta$  coordinate in (4.2) gives (4.1). Since the integral curves of  $X_H$  (with  $H$  the Hamiltonian of the system) define the time evolution of the system, it is interesting to investigate the integral curves of  $V_H$  on  $Y$ . If  $(\mathbf{r}(t), \mathbf{p}(t))$  denotes an integral curve of  $X_H$ , then one easily verifies that the integral curves of  $V_H$  are given by  $(\mathbf{r}(t), \mathbf{p}(t), e^{i\theta(t)})$  with:

$$e^{i\theta}(t) = e^{i\theta(t_0)} \exp\left(-i \int_{t_0}^t [\mathbf{p}(s) \cdot \frac{\partial H}{\partial \mathbf{p}}(\mathbf{r}(s), \mathbf{p}(s)) - H(\mathbf{r}(s), \mathbf{p}(s))] ds\right). \quad (4.3)$$

Physicists will recognise the integrand in the exponential as the Lagrangian  $L$  of the system (see the end of §2.3) and hence they will recognise the integral itself as the action and the complete exponential as the phase factor which plays the fundamental role in the Feynman path integral.

We see that the introduction of the space  $Y = M \times U(1)$ , in which the states of the physical system are described by circles, has two consequences. In the first place we now can represent the Poisson algebra (functions on  $M$ ) injectively by vector fields on  $Y$  which project on the Hamiltonian vector fields on  $M$ ; in the second place we find that in the time evolution of the phase factor  $e^{i\theta}$  is given by the phase factor of the Feynman path integral.

For the moment we stay in the realm of mathematics where two important questions arise: 'what are the relevant features of this construction' and 'can we obtain similar results for an arbitrary symplectic manifold'? Prequantization (or the prequantization formulation of classical mechanics) is a theory which gives an affirmative answer to the second question, of course after specifying what the 'relevant' features are. In the first place the important feature of  $Y$  is that it carries a 1-form  $\alpha$  given by:

$$\alpha = \mathbf{p} \cdot d\mathbf{r} + d\theta \quad (4.4)$$

which satisfies  $d\alpha = \omega$  and which turns  $Y$  into a principal  $U(1)$  fibre bundle over  $M$  with connection  $\alpha$  and curvature  $\omega$ . According to this point of view the vector fields  $V_f$  are the unique vector fields on  $Y$  which satisfy the conditions:

$$\left. \begin{aligned} \pi^* V_f &= X_f && \text{(i.e. } V_f \text{ projects onto } X_f) \\ \alpha(V_f) &= f \end{aligned} \right\} \quad (4.5)$$

It turns out that one can find for all symplectic manifolds  $(M, \omega)$  (except for some nasty cases to be defined below) a principal fibre bundle  $Y$  over  $M$  with connection  $\alpha$  such that the structure group is the circle and that the curvature is  $\omega$ . If such a  $(Y, \alpha)$  exists, one can construct a unique injective Lie algebra morphism  $f \rightarrow V_f$  from the Poisson algebra to vector fields on  $Y$  satisfying (4.5). Although the introduction of the 1-form  $\alpha$  seems quite ad hoc, it should be mentioned that it appears quite naturally when one looks for injective representations of the Poisson algebra as vector fields on a bundle over  $M$  which project to the (non-injective) representation by the Hamiltonian vector fields on  $M$ .

Before going on to the next subsection which describes some of the consequences, we have to mention some details concerning the construction of the bundle  $Y$  and its (connection) form  $\alpha$ . When the symplectic manifold  $(M, \omega)$  is given, one can define a subgroup  $\text{Per}(\omega)$  of  $\mathbb{R}$  called the group of periods of the symplectic form  $\omega$ . There exists a bundle  $(Y, \alpha)$  over  $M$  with the desired properties if and only if  $\text{Per}(\omega)$  is discrete in  $\mathbb{R}$ . If  $\text{Per}(\omega)$  is discrete, then there exists a unique non-negative real number  $p_{\text{er}}$  called the generator of  $\text{Per}(\omega)$ , such that  $\text{Per}(\omega) = p_{\text{er}} \mathbb{Z} \subset \mathbb{R}$ . It then follows that there exists a principal fibre bundle  $Y$  over  $M$  with structure group  $\mathbb{R}/\text{Per}(\omega) (\cong \mathbb{R} \text{ modulo } p_{\text{er}} \cong \text{a circle with radius } p_{\text{er}}/2\pi)$  and a connection form  $\alpha$  with  $d\alpha = \omega$ . If  $p_{\text{er}} = 0$ , then the space  $Y$  constructed in this way is not a circle bundle but an  $\mathbb{R}$  bundle  $(\mathbb{R}/\{0\} = \mathbb{R})$ ; however, one can wind this real line over any period to obtain a circle and ‘hence’ a circle bundle with connection  $\alpha$  satisfying  $d\alpha = \omega$ . Since all circle groups are isomorphic to  $U(1)$ , it follows that we have obtained a space  $Y$  and a projection  $\pi: Y \rightarrow M$  such that the inverse image of a point  $m \in M$  is a circle of ‘phase factors’.

In the general case the bundle  $(Y, \alpha)$  constructed in this way is not unique; it is unique if  $M$  is simply connected. If  $(Y, \alpha)$  is not unique, one can sometimes use the non-uniqueness to explain effects in physics which can not be understood at the level of the phase space  $M$  (e.g. the Bohm-Aharonov experiment, see [Wo]). If the phase space  $M$  is the cotangent bundle of some configuration space  $Q$ , i.e.  $M = T^*Q$ , with its canonical symplectic form  $\omega$  as defined in (2.6) then  $p_{\text{er}} = 0$  and  $Y = T^*Q \times U(1)$  is a possible choice for  $Y$  as in the example of  $M = \mathbb{R}^{2n}$ ; it is the unique choice if  $Q$  is simply connected.

Finally one should know that one can always choose a local trivialisation (local gauge) such that the 1-form  $\alpha$  looks locally like (4.4) where  $(\mathbf{r}, \mathbf{p})$  are (local) coordinates on  $M$  and  $\theta$  a coordinate on the circle  $\mathbb{R} \text{ mod } p_{\text{er}}$  (hence not always modulo  $2\pi$ !). The choice of a different trivialisation corresponds to a local gauge transformation, which affects the local expression of  $\alpha$  and hence the local expression of  $V_f$ . However, the changes are such that the Lagrangian  $L = \mathbf{p} \cdot \partial H / \partial \mathbf{p} - H$  in (4.3) is changed by a ‘total time derivative’, i.e. the integral in (4.3) is changed by a boundary term, which can be interpreted as a change of the coordinate  $\theta$ .

#### 4.2. Analogies between classical and quantum mechanics

This subsection will contain only statements of results; readers unfamiliar with the notion of principal fibre bundle and connection should always keep in mind the example of  $M = \mathbb{R}^{2n}$ ,  $Y = \mathbb{R}^{2n} \times U(1)$  (and  $\alpha = \mathbf{p} \cdot d\mathbf{r} + d\theta$ ). First of all some terminology: the bundle  $Y$  is called the *prequantum bundle* over the symplectic manifold  $M$ ; the reason for this name will become clear in §5. In the second place we need some facts about the prequantum bundle  $(Y, \alpha)$  over  $(M, \omega)$ . It is well known that the flow of a Hamiltonian vector field  $X_f$  on  $M$  preserves the symplectic form  $\omega$  (which implies Liouville's theorem stating that a canonical transformation preserves the so called Liouville measure on the phase space). From the definition (4.5) of the vector fields  $V_f$  one can deduce that the flow of a vector field  $V_f$  on  $Y$  preserves the 1-form  $\alpha$ . With these preparations we can state the analogies between classical and quantum mechanics.

##### Analogy 1

In classical mechanics each state of the system is represented in  $M$  by a point and in  $Y$  by a circle; in quantum mechanics a state is represented in  $\mathbb{P}\mathcal{H}$  by a point and in  $S\mathcal{H}$  by a circle.

##### Analogy 2

In quantum mechanics the projective Hilbert space  $\mathbb{P}\mathcal{H}$  is an (infinite dimensional) symplectic manifold. If we construct the prequantum bundle  $Y$  over this symplectic manifold  $\mathbb{P}\mathcal{H}$ , we obtain the unit sphere  $S\mathcal{H}$  and in this case the prequantum bundle is unique. Moreover, the dynamics on  $\mathbb{P}\mathcal{H}$  is given by the Hamiltonian vector field  $X_{E(\mathbb{H})}$  associated to the Hamiltonian  $E(\mathbb{H})$  and the unique lift  $V_{E(\mathbb{H})}$  defined by (4.5) to the prequantum bundle  $Y = S\mathcal{H}$  is the vector field  $V_{\mathbb{H}}$  on  $S\mathcal{H}$  defined in (3.6).

##### Symmetry transformations

According to Wigner, a symmetry of the quantum description of a physical system is a bijection of the projective Hilbert space  $\mathbb{P}\mathcal{H}$ , i.e. a bijective mapping from states to states, which moreover preserves the transition probabilities, i.e. which preserves the form  $P$  on  $\mathbb{P}\mathcal{H}$  defined by:

$$P: \mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H} \rightarrow \mathbb{R}_{\geq 0}, \quad P(\pi\psi, \pi\chi) = \frac{|\langle \psi, \chi \rangle|^2}{\|\psi\|^2 \|\chi\|^2} \quad (4.6)$$

where  $\langle, \rangle$  denotes the inner product on  $\mathcal{H}$  and  $\|\psi\|^2 = \langle \psi, \psi \rangle$ . Using this definition of a symmetry Wigner showed that for each symmetry  $g$  there exists either a unitary or an anti unitary operator  $U(g)$  on the Hilbert space  $\mathcal{H}$  such that  $U(g)$  induces  $g$ , i.e.

$$\pi(U(g)\psi) = g(\pi\psi).$$

Moreover, he showed that if  $U'(g)$  is another (anti) unitary operator on  $\mathcal{H}$  which induces  $g$ , then  $U'(g)$  differs a phase factor from  $U(g)$ :  $U'(g) = e^{i\theta} U(g)$ . It follows that if  $G$  is a connected Lie group of symmetries of the quantum



system, then each symmetry can be represented by a unitary transformation of the Hilbert space  $\mathcal{H}$ . It does *not* imply that the group  $G$  can be represented as a *group* of unitary operators on  $\mathcal{H}$ , because we only know that  $U(g)U(h) = e^{i\theta}U(gh)$  for some phase factor  $e^{i\theta}$  depending on  $g$  and  $h$ , but this phase factor is in general different from 1.

The best one can do in the general case is to construct a group  $G'$  of unitary transformations of the Hilbert space  $\mathcal{H}$  together with a surjective group homomorphism  $\pi': G' \rightarrow G$  which has the following properties. In the first place the action of  $G'$  on  $\mathcal{H}$  induces the action of  $G$  on  $\mathbb{P}\mathcal{H}$ , i.e.

$$\pi'(g')(\pi(\psi)) = \pi(g'\psi) \quad g' \in G' \text{ and } \psi \in \mathcal{H} \setminus \{0\}$$

and in the second place the kernel  $\ker(\pi')$  of  $\pi'$  is isomorphic to  $U(1)$  and commutes with all elements of  $G'$  (in such a situation one calls  $G'$  a *central extension* of  $G$  by  $U(1)$ ). A different way to visualise the group  $G'$  is to say that  $G'$  consists of *all* possible choices  $U(g)$  for  $g \in G$  (which obviously has a homomorphism onto  $G(U(g) \rightarrow g)$  whose kernel is isomorphic to  $U(1)$  (the freedom in  $U(g)$  is a phase factor). It should be noted that this group  $G'$  is uniquely determined by the action of the group  $G$  on  $\mathbb{P}\mathcal{H}$ .

It sometimes happens that  $G'$  equals  $G \times U(1)$  as product of groups in which case one obviously can represent  $G$  as a group of unitary transformation on  $\mathcal{H}$  by identifying  $G$  with the subgroup  $G \times \{1\}$  of  $G \times U(1)$ ; in the other cases  $G$  can *not* be represented as a group of unitary transformations on  $\mathcal{H}$ . Examples of these possibilities are abundant in physics: the rotation group  $SO(3)$  is a symmetry group at the level of the projective Hilbert space; if the system consists of particles with half-integer spin, then the central extension  $G'$  is not the direct product of  $SO(3)$  with  $U(1)$  so  $SO(3)$  can not be represented as a group of unitary transformations of  $\mathcal{H}$ . If we use the double covering group  $SU(2)$  of  $SO(3)$  instead of  $SO(3)$  itself, then the extension  $G'$  is the trivial product  $SU(2) \times U(1)$  hence  $SU(2)$  can be represented as a group of unitary transformations of  $\mathcal{H}$ . Physicists conclude that  $SO(3)$  is not the correct symmetry group of the quantum description, but that  $SU(2)$  is the symmetry group of the quantum description which represents the rotation invariance. Exactly the same situation occurs for (the connected component of) the Lorentz group  $L$ : for  $L$  the central extension  $G'$  is not the trivial product but for its double covering  $SL(2, \mathbb{C})$  the central extension is the trivial product. Again one concludes that  $SL(2, \mathbb{C})$  is the correct symmetry group of the quantum description which represents the Lorentz invariance. In these examples it turns out that if we replace the symmetry group by its simply connected covering, then we get a representation of this covering as a group of unitary transformations on  $\mathcal{H}$ . That this is not the general idea is clear when we study the group  $\mathbb{R}^{2n}$  of translations in position and momentum applied to the quantum description of a particle with classical phase space  $\mathbb{R}^{2n}$ . This group is already simply connected, but the central extension  $G'$  is not the trivial product but the Heisenberg group (in physics this group is sometimes called the Weyl-Wigner group). It turns out that *if*  $G$  is a simply connected *and* semi simple Lie group, *then* the central extension  $G'$  is the trivial product. Hence if these two conditions are

satisfied, then we know in advance that  $G$  can be represented as a group of unitary transformations on  $\mathcal{H}$ .

One final remark before we state the third analogy between classical and quantum mechanics. At the level of the projective Hilbert space  $\mathbb{P}\mathcal{H}$ , a symmetry group  $G$  is a group of bijections which preserve the form  $P$  defined in (4.6). At the level of the Hilbert space such a symmetry group is represented by a central  $U(1)$  extension  $G'$  of bijections which preserve the inner product  $\langle, \rangle$  on  $\mathcal{H}$ . Since  $G'$  preserves the inner product, it preserves the unit sphere  $S\mathcal{H}$ , hence we can interpret  $G'$  as a group of bijections of the unit sphere  $S\mathcal{H}$  which preserve the form  $\langle, \rangle$  on  $S\mathcal{H}$ .

### *Analogy 3*

A symmetry group of the quantum description of a system is a group  $G$  of bijections of  $\mathbb{P}\mathcal{H}$  which preserve the form  $P$ . One can always find a group  $G'$  of bijections of  $S\mathcal{H}$  which preserve  $\langle, \rangle$  such that each symmetry  $g \in G$  is represented by a circle (of phase factors) in  $G'$  and the action of this circle on  $S\mathcal{H}$  induces the action of  $g$  on  $\mathbb{P}\mathcal{H}$ .

In classical mechanics ‘exactly’ the same situation occurs. A symmetry group of the classical description of a system is a group  $G$  of bijections of the phase space  $M$  which preserve the symplectic form  $\omega$  (Liouville). One can ‘always’ (with the exception of some topologically nasty cases) find a group  $G'$  of bijections of the prequantum bundle  $Y$  which preserve the 1-form  $\alpha$  such that each symmetry  $g \in G$  is represented by a circle (of phase factors) in  $G'$  and the action of this circle on  $Y$  induces the action of  $g$  on  $M$ .

Moreover, if a physical system is described in both classical mechanics and quantum mechanics, and if we assume that a group appears as a symmetry group for both descriptions, then the same extensions appear. If the classical system described particles of half-integer spin, then the extension of  $SO(3)$  is not trivial; the extension of  $SU(2)$  is always trivial because it is a simply connected, semi simple Lie group. If the translations in position and momentum form a symmetry group, then the associated extension (which acts on the prequantum bundle  $Y$  in the prequantization description) is the Heisenberg group.

## 5. GEOMETRIC QUANTIZATION

### *5.1. Quantization in general*

When physicists speak about quantization they mean a loosely process which tells them how to obtain the mathematical ingredients of the quantum description of a physical system (§3) if the mathematical ingredients of the classical description (§2) are known. The idea that such a procedure should exist can be traced back to a remark of Dirac who observed that there exists a remarkable resemblance between the Poisson brackets of functions on the (classical) phase space and the commutator of the corresponding quantum observables (self adjoint operators on  $\mathcal{H}$ ). To be more specific, he observed that if  $f$  and  $g$  are (real) functions on the (classical) phase space representing observable quantities (i.e.  $f(m)$  represents the result of measuring  $f$  when the system is described

by the point  $m$  of the phase space) and if  $\underline{f}$  and  $\underline{g}$  are the self adjoint operators on  $\mathcal{H}$  representing the same observable quantities in the quantum description, then the self adjoint operator  $\underline{h}$  which represents the (classical) observable  $h = \{f, g\}_{\text{Poisson}}$  is equal to  $(i/\hbar)[\underline{f}, \underline{g}]$ . To state this property in a more formal way we write  $O(f)$  for the self adjoint operator  $\underline{f}$  in  $\mathcal{H}$  which represents the observable  $f$ . With this notation the observation of Dirac can be stated as:

$$[O(f), O(g)] = -i\hbar O(\{f, g\}).$$

When we apply this 'rule' to the position and momentum observables  $p$  and  $r$  on the phase space  $\mathbb{R}^2$  we find the well known canonical commutation relation:

$$[O(p), O(r)] = -i\hbar Id, \tag{5.1}$$

where we have made the additional assumption that a constant observable (a function on the phase space which takes a constant value) should be represented by the same constant times the identity operator on  $\mathcal{H}$ .

In the physical literature several (different) quantization procedures are known, e.g. canonical quantization, Weyl-Wigner quantization, quantization by Feynman-path integrals, stochastic quantization, quantization by \*-products and geometric quantization. Although the starting points of these quantization procedures are quite different, the first test for these procedures always is to see whether the canonical commutation relation (5.1) is satisfied if the phase space is  $\mathbb{R}^2$ . In this aspect canonical quantization is the most drastic: it takes the canonical commutation relation (5.1) as basic axiom for the quantization. We now intend to formalise the notion of quantization with the canonical commutation relations and Dirac's remark in mind. It should be a procedure which has as input the phase space  $(M, \omega)$  of a system in the symplectic formulation and as output a Hilbert space  $\mathcal{H}$  together with a map  $O$  which assigns to a real valued function  $f$  on  $M$  a self adjoint operator  $O(f) \equiv \underline{f}$  on  $\mathcal{H}$ , such that certain conditions are satisfied. In the first place, the physical contents should be the same, i.e.  $M$  and  $\mathcal{H}$  describe the same physical system and if the function  $f$  on  $M$  represents an observable quantity, then  $O(f)$  should represent the same physical quantity. In the second place the map  $O$  should satisfy certain desirable conditions  $Q(i) \dots Q(v)$  listed below.

$$\left. \begin{array}{ll} Q(i) & O(f+g) = O(f) + O(g) \\ Q(ii) & O(\lambda f) = \lambda O(f) \ (\lambda \in \mathbb{R}) \end{array} \right\} \Leftrightarrow O \text{ is } \mathbb{R} \text{-linear}$$

where  $f+g$  is the pointwise addition of functions. I do not know of a physical motivation for this linearity condition, but it is certainly desirable from the computational point of view. However, even this linearity condition raises questions: a priori it is not guaranteed that the domains of  $O(f)$  and  $O(g)$  are the same, so in condition  $Q(i)$  one has to be careful with the domains of the

operators. Another desideratum is that the constant observable 1 is represented by the identity operator:

$$Q(iii) \quad O(1) = Id_{\mathfrak{H}}$$

which represents the idea that if a measurement yields always 1 in the classical description, then it should yield 1 in the quantum description too (see §3.1). According to Dirac's remark one also would expect  $O$  to be a Lie algebra morphism:

$$Q(iv) \quad [O(f), O(g)] = -i\hbar O(\{f, g\}).$$

The final condition to be imposed on a quantization procedure should be that it yields the well known and well tested Schrödinger quantization in the case of  $M = \mathbb{R}^{2n}$ , i.e.  $\mathfrak{H} = L^2(\mathbb{R}^n)$  (the square integrable (complex) functions of  $\mathbf{r} \in \mathbb{R}^n$ ) together with the following assignments for some operators:  $O(\mathbf{r}) = \mathbf{r}$  (pointwise multiplication) and  $O(\mathbf{p}) = -i\hbar \partial / \partial \mathbf{r}$  (differentiation with respect to  $\mathbf{r}$ ). In view of a theorem of Stone and Von Neumann this condition can be reformulated as:

$$Q(v) \quad \begin{cases} \mathfrak{H} \text{ is irreducible} \\ \text{the action of a} \\ \text{complete set of canonical coordinates} \end{cases}$$

(this reformulation by means of Stone-Von Neumann is not quite correct, but for the sake of simplicity we ignore the technical details). Condition  $Q(v)$  has a clear physical motivation: the (abelian) group of translations in position and momentum  $\mathbb{R}^{2n}$  acts transitively on the phase space  $M = \mathbb{R}^{2n}$ , i.e. there exists no proper submanifold of  $M$  which is invariant under the action of the group of translations  $\mathbb{R}^{2n}$ .  $Q(v)$  is a different formulation of the same idea: there should not exist a proper subspace of  $\mathfrak{H}$  which is invariant under the action of the position and momentum operators (because the canonical coordinates are position and momentum).

Unfortunately, condition  $Q(v)$  does not make sense for a general phase space, i.e. a symplectic manifold, because in general there do not exist *global* canonical coordinates (although Darboux's theorem tells us that *locally* there always exist canonical coordinates). With the motivation for  $Q(v)$  in mind, we can reformulate  $Q(v)$  in such a way that it makes sense for a larger class of symplectic manifolds. Suppose  $G$  is a symmetry group of the symplectic manifold  $(M, \omega)$ , i.e. each diffeomorphism  $g \in G$  of  $M$  leaves  $\omega$  invariant (see also analogy 3, §4.2). If we assume  $G$  to be a symmetry group of the quantum description too, then it follows from §4.2 that a central extension  $G'$  of  $G$  acts as a group of unitary transformations on the Hilbert space  $\mathfrak{H}$ . If we now suppose that  $G$  acts transitively on  $M$ , as is the case with  $M = G = \mathbb{R}^{2n}$ , then we can reformulate  $Q(v)$  as  $Q(v)'$ :

$$Q(v)' \quad \begin{cases} \mathfrak{H} \text{ is an irreducible representation} \\ \text{for (a central extension of) } G \end{cases}$$

In this way  $Q(v)'$  is applicable to a large class of symplectic manifolds (and if we go to the category of diffeological manifolds as developed by J.-M. Souriau

[So3],  $Q(v)$  is applicable to all symplectic manifolds).

However, Van Hove has shown in [vH] (see also [Ab&Ma]) that even for the phase space  $\mathbb{R}^{2n}$  the five conditions  $Q(i)–Q(v)$  are incompatible! For  $M = \mathbb{R}^{2n}$  there does not exist a map  $O$  from all (smooth) functions  $f$  on  $M$  to self adjoint operators on any Hilbert space  $\mathfrak{H}$  such that the conditions  $Q(i)–Q(v)$  are satisfied. That this could happen was already felt by Dirac himself because he phrases his remark which led to canonical quantization as: ‘... the quantum-brackets, or at any rate the simpler ones, have the same values as the corresponding classical Poisson brackets’ [Di, p87]. What physicists usually do is to weaken condition  $Q(iv)$  by requiring that  $Q(iv)$  holds only for a certain subset of all observables (but which one is in general not specified).

### 5.2. The first step in the geometric quantization procedure

In the previous section we have seen that the quite natural conditions  $Q(i)–Q(v)$  cannot be used as an axiomatic basic for quantization because they are contradictory. Geometric quantization is a quantization procedure which tries to define in an intrinsic way the Hilbert space  $\mathfrak{H}$  and the map  $O$  when the symplectic manifold  $(M, \omega)$  is given, such that  $Q(i)–Q(iii)$  are always satisfied, such that  $Q(iv)$  is satisfied for a (well defined) subset of observables and such that  $Q(v)$  holds whenever applicable. Moreover, when both the classical and the quantum description of a physical system are known, it tries to obtain the given quantum description from the classical description. It should be said up to now geometric quantization is not completely succesful in the realisation of this program (e.g. see [Du]).

To give the reader an idea how the geometric quantization procedure works, we take the well known phase space  $M = \mathbb{R}^{2n}$  with its canonical symplectic form  $\omega$  (formula (2.6)) as an example. The Liouville measure  $\epsilon^L = dp_1 \dots dp_n dr_1 \dots dr_n$  on  $M$  gives us in a canonical way a Hilbert space:  $\mathfrak{H}_1 = L^2(\mathbb{R}^{2n}, \epsilon_L)$ , i.e. the space of square integrable (complex) functions on  $M$ . Using the Hamiltonian vector field  $X_f$  associated to (real) functions we can define (self adjoint) operators  $O_1(f)$  on  $\mathfrak{H}_1$  by:

$$O_1(f) = -i\hbar X_f.$$

One now can verify that the map  $O_1$  satisfies the conditions  $Q(i)$ ,  $Q(ii)$  and  $Q(iv)$  (because the map  $f \rightarrow X_f$  is a Lie algebra morphism) but it does not satisfy condition  $Q(iii)$  since the Hamiltonian vector field of a constant function is zero.

To find a Hilbert space and a map  $O$  for which  $Q(iii)$  is satisfied, we can use the prequantum bundle  $(Y, \alpha)$  with  $Y = \mathbb{R}^{2n} \times U(1)$  over the symplectic manifold  $(M, \omega)$ . On  $Y$  we also have a canonical measure  $\epsilon_Y = dp_1 \dots dp_n dr_1 \dots dr_n d\theta$  (with  $e^{i\theta}$  the element of  $U(1)$ ), so we can define a Hilbert space  $\mathfrak{H}_2 = L^2(\mathbb{R}^{2n} \times U(1), \epsilon_Y)$ , i.e. the space of square integrable (complex) functions on  $Y$ . Using the injective representation  $V$  of the Poisson algebra (formula (4.5)) we can define operators  $O_2(f)$  on  $\mathfrak{H}_2$  by:

$$O_2(f) = -i\hbar V_f.$$

Again, one easily verify that  $O_2$  satisfies conditions  $Q(i)$ ,  $Q(ii)$  and  $Q(iv)$ , but as before  $O_2$  does not satisfy  $Q(iii)$ . Although the operator  $O_2(1)$  is not the zero operator on  $\mathfrak{K}_2$ , neither is it the identity operator (it is  $-i\hbar[\partial/\partial\theta]$ ). We now define  $\mathfrak{K}_3$  as the subspace of  $\mathfrak{K}_2$  on which  $O_2(1)$  is the identity operator; one deduces from the expression of  $O_2(1)$  that  $\mathfrak{K}_3$  is given by:

$$\mathfrak{K}_3 = \{f \in \mathfrak{K}_2 | f(\mathbf{r}, \mathbf{p}, \theta) = f(\mathbf{r}, \mathbf{p})e^{i\theta/\hbar}\}. \quad (5.2)$$

From the expression of the vector fields  $V_f$  (formula (4.2)) one deduces that the Hilbert space  $\mathfrak{K}_3$  is invariant under the action of the operators  $O_2(f)$ , so we can define  $O_3$  as the same map as  $O_2$  (but in this case the operators have to be seen as operators on  $\mathfrak{K}_3$ ).

Table

$\mathfrak{K}_1 =$ functions on $M$	$O_1(f) = -i\hbar X_f$
$\mathfrak{K}_2 =$ functions on $Y$	$O_2(f) = -i\hbar V_f$
$\mathfrak{K}_3 = \psi \in \mathfrak{K}_2: O_2(1)\psi = \psi$	$O_3(f) = O_{sub2}(f) = -i\hbar V_f$

What we gave got now is a Hilbert space  $\mathfrak{K}_3$  and a map  $O_3$  which satisfies the conditions  $Q(i) - Q(iv)$ ; our only worry is condition  $Q(v)'$ . However, before we investigate whether  $Q(v)'$  is satisfied or not (it is not), we will say a few words about the case of a general phase space. We have said already that if  $(M, \omega)$  is any symplectic space, then there exists (nearly always) a prequantum bundle  $(Y, \alpha)$  over  $(M, \omega)$  and an injective representation  $V$  of the Poisson algebra as vector fields on  $Y$ . In such a case one can always define the analogue of  $\mathfrak{K}_2$  and the map  $O_2$  ( $Y$  has a canonical measure defined by the 1-form  $\alpha$ ), but sometimes the subspace  $\mathfrak{K}_3$  consists of the zero vector only. Whether  $\mathfrak{K}_3$  is  $\{0\}$  or not depends upon the value of the number  $p_{er}$  which is associated to the symplectic manifold  $(M, \omega)$  (see the end of §4.1):  $\mathfrak{K}_3 \neq \{0\}$  if and only if  $p_{er}$  is an integer multiple of  $2\pi\hbar$ . An 'explanation' of this condition can be found in studying formula (5.2). We know from the discussion at the end of §4.1 that the coordinate  $\theta$  should be taken modulo  $p_{er}$ , hence  $\theta=0$  and  $\theta=p_{er}$  indicate the same point, which in turn 'implies' that  $\exp(ip_{er}/\hbar) = 1$ .

This condition on the symplectic manifold ( $p_{er}$  an integer multiple of  $2\pi\hbar$ ) can be thought of as a quantization condition: it restricts the possible phase spaces for which this method works. To show that this condition has a meaning in physics, we first memorate that if  $(M, \omega)$  is a cotangent bundle with its canonical symplectic form, then  $p_{er}$  is zero, so the quantization condition is trivially satisfied and a pair  $(\mathfrak{K}_3, O_3)$  with the properties  $Q(i) - Q(iv)$  always exists. Secondly, if  $(M, \omega)$  is the symplectic space which represents in the classical formalism a particle with spin, i.e.  $M = S^2$  and  $\omega = \lambda \sin\theta d\theta \wedge d\phi$  (see the end of §2.5), then  $p_{er} = 4\pi\lambda$  and the quantization conditions becomes the well known quantization of spin in quantum mechanics  $\lambda = n\hbar/2$  (with  $n \in \mathbb{Z}$ ). One of the conclusions of these observations could be that this procedure, i.e. the construction of a pair  $(\mathfrak{K}_3, O_3)$  by means of the prequantum bundle  $Y$ , gives

us relevant information on the quantum behaviour of the physical system when we only know its classical (symplectic) description. This consideration plus the fact that  $(\mathcal{K}_3, O_3)$  satisfies  $Q(i) \dots Q(iv)$  explains the names prequantization and prequantum bundle:  $(Y, \alpha)$  and  $(\mathcal{K}_3, O_3)$  can be seen as a first step towards a quantum description.

Let us now return to condition  $Q(v)$  and our example  $M = \mathbb{R}^{2n}$ . Since we know the dependence of functions in  $\mathcal{K}_3$  on the coordinate  $\theta$ , we can identify  $\mathcal{K}_3$  with  $L^2(\mathbb{R}^{2n})$ . Using formulas (4.2) and (5.2) we find that  $O_3(f)$  can be expressed by:

$$O_3(f) = -i\hbar \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{r}} + i\hbar \frac{\partial f}{\partial \mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{p}} + f - \mathbf{p} \cdot \frac{\partial f}{\partial \mathbf{p}}$$

from which one deduces that  $O_3(\mathbf{p}) = -i\hbar \partial / \partial \mathbf{r}$ , which is indeed differentiation with respect to  $\mathbf{r}$  and that  $O_3(\mathbf{r}) = \mathbf{r} + i\hbar \partial / \partial \mathbf{p}$ , which is not multiplication by  $\mathbf{r}$ . That we do not obtain the usual Schrödinger quantization (see §5.1) is in complete agreement with Van Hove's result that the conditions  $Q(i) - Q(v)$  are incompatible. In fact Van Hove has indicated in [vH] a proper subspace of  $\mathcal{K}_3$  which is invariant under the action of the operators  $O_3(\mathbf{r})$  and  $O_3(\mathbf{p})$ .

### 5.3. THE IDEA OF A POLARIZATION AND PAIRING

The main idea to obtain (in an intrinsic way) the Schrödinger quantization from the constructions described in the previous subsection, the idea which is the heart of the geometric quantization procedure, is the idea of a polarization. Since the technicalities of the construction with polarizations are beyond the scope of this paper, this section will be brief and it will only outline the main idea and mention the problems which one encounters. The first observation one should make is that *if* we restrict  $\mathcal{K}_3$  to functions which depend only on the coordinates  $\mathbf{r}$  *and if* we integrate these functions *only* over  $\mathbf{r}$ , *then* the operators  $O_3(\mathbf{r})$  and  $O_3(\mathbf{p})$  are the correct operators on  $L^2(\mathbb{R}^n)$  for the Schrödinger quantization. The problem then is that for a general observable  $f$  (e.g. the kinetic energy  $|\mathbf{p}|^2/2$ ) the operator  $O_3(f)$  does not have any meaning on this 'subspace' of  $\mathcal{K}_3$ ; only for functions which are at most linear in  $\mathbf{p}$  makes  $O_3(f)$  any sense on the space of square integrable functions in  $\mathbf{r}$ .

Nevertheless, it is this idea which is generalised by a polarization. With regard to polarizations, the main property of the coordinates  $r_1 \dots r_n$  is that they constitute a maximal set of independent functions in involution, i.e. the differentials of them are everywhere linearly independent, the Poisson bracket of any pair of them is zero and the set is maximal with respect to the previous two properties. Roughly speaking, a polarization  $F$  on a symplectic manifold is a set of (complex) functions  $f_1 \dots f_n$  which is a maximal set of independent functions in involution; these functions need not be defined globally, but in every neighbourhood such a system should exist and they should satisfy compatibility conditions on the intersection of two such neighbourhoods. One can deduce that the number of independent functions in a polarization  $F$  is always half the dimension of the symplectic manifold  $M$  and that if the functions are real there exist functions  $g_1 \dots g_n$  such that the set of functions  $f_1 \dots f_n, g_1 \dots g_n$

constitutes a system of canonical coordinates on the symplectic manifold. One then constructs a Hilbert space  $\mathcal{H}_{GQ}$  ( $GQ$  to indicate *Geometric Quantization*) which consists of functions on  $Y$  on which  $O_3(1)$  is the identity operator (i.e. which depend in a specified way on the coordinate  $\theta$  in the fibre of the bundle  $Y$ ) and which depend only on the ‘coordinates’  $f_1 \dots f_n$ ; the norm of such functions is defined by integrating over the ‘coordinates’  $f_1 \dots f_n$ .

The main technical problem with this approach is the question how to define ‘integrating over the coordinates  $f_1 \dots f_n$ ’ for a general polarization  $F$ . The first solution was the idea of half-densities (something like square roots of measures), but this did not give the correct quantization of the harmonic oscillator. The second solution is the idea of half-forms (something like square roots of volume forms) which gives the correct results for the harmonic oscillator. However, there remain systems for which even the solution with half-form does not yield the correct answers (e.g. see [Du]), but it is the best solution till now. The map  $O_{GQ}$  which gives us the operators on  $\mathcal{H}_{GQ}$  associated to classical observables is derived from the map  $O_3$  but, as we have seen in our example, the domain of  $O_{GQ}$  is a rather small set of observables which depends strongly upon the choice of the polarization  $F$ . On this small domain the map  $O_{GQ}$  satisfies the conditions  $Q(i) - Q(iv)$ . In the case of  $M = \mathbb{R}^{2n}$  and the polarization defined by the position coordinates  $r$  it also satisfies  $Q(v)$ ; for a certain class of Lie Groups  $G$  one can show that if such a Lie group  $G$  acts transitively on  $M$  as diffeomorphisms which conserve the symplectic form  $\omega$ , then the modified condition  $Q(v)$  is satisfied (for mathematicians: this is essentially the Borel-Weil-Bott theorem on irreducible representations of compact simply connected Lie groups).

Let us give some examples of polarizations and the associated Hilbert spaces for our example  $M = \mathbb{R}^{2n}$ . As already said  $F_r = \{r_1 \dots r_n\}$  is a polarization and the associated Hilbert space  $\mathcal{H}_r$  is the space of square integrable functions of the coordinates  $r$  with respect to the Lebesgue measure on  $\mathbb{R}^n$ . Another polarization is given by the momentum coordinates:  $F_p = \{p_1 \dots p_n\}$  for which the associated Hilbert space  $\mathcal{H}_p$  is the space of square integrable functions of the coordinates  $p$  with respect to the Lebesgue measure (physicists will recognise these two Hilbert spaces as the position and momentum representations of the Schrödinger quantization). Yet another polarization is given by the complex functions  $z_j$  defined by  $z_j = p_j + ir_j$ :  $F_z = \{z_1 \dots z_n\}$ ; in this case the associated Hilbert space  $\mathcal{H}_z$  is the space of holomorphic functions in  $z$  (i.e. functions independent of the coordinates  $z_j^*$  which are the complex conjugates of the  $z_j$ ) which are square integrable with respect to the Gaussian measure  $\exp(-\sum_j |z_j|^2)$  on  $\mathbb{R}^{2n} = \mathbb{C}^n$ ; this is called the Bargmann representation of the Schrödinger quantization.

It is known that there exist unitary equivalences between the three Hilbert spaces described above (between  $\mathcal{H}_r$  and  $\mathcal{H}_p$  it is the Fourier transform and between  $\mathcal{H}_r$  and  $\mathcal{H}_z$  it is the Bargmann transform), so the natural question is whether we can ‘always’ construct in an intrinsic way a unitary equivalence between the Hilbert spaces associated to different polarizations on the same symplectic manifold. Up till now the answer is sometimes yes and sometimes



no. Under certain conditions on two polarizations  $F_1$  and  $F_2$  on a given symplectic manifold  $(M, \omega)$  with associated Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$  one can define a map  $\Psi: \mathcal{H}_1 \times \mathcal{H}_2 \rightarrow \mathbb{C}$  which is linear in the second coordinate and anti linear in the first. This map is called the pairing and is defined by means of a kernel function, the so called BKS-kernel after Blattner, Kostant and Sternberg.  $\Psi$  is defined by integration of this kernel function over (not always all coordinates of)  $M$ . One hopes that such a pairing defines bijective maps:  $A: \mathcal{H}_1 \rightarrow \mathcal{H}_2$  and  $B: \mathcal{H}_2 \rightarrow \mathcal{H}_1$  such that:

$$\langle \psi_1, B\psi_2 \rangle_1 = \Psi(\psi_1, \psi_2) = \langle A\psi_1, \psi_2 \rangle_2$$

where  $\langle \cdot, \cdot \rangle_j$  is the inner product on  $\mathcal{H}_j$  and moreover, one hopes that  $A$  and  $B$  are unitary. It turns out that the pairing defined by the BKS-kernel applied to the case of the polarizations  $F_r, F_p$  and  $F_z$  on  $\mathbb{R}^{2n}$  defines indeed unitary maps which are the already mentioned unitary equivalences between  $\mathcal{H}_r, \mathcal{H}_p$  and  $\mathcal{H}_z$ . Unfortunately no general theorem is known under which circumstances this pairing defines a unitary equivalence between the Hilbert spaces associated to different polarizations; there exists an example in which the pairing defines a dilatation and there exists an example in which the pairing defines bijective maps  $A$  and  $B$  which are neither unitary nor dilatations.

We conclude this section with three remarks.

- 1) The inner product on a Hilbert space  $\mathcal{H}_{GQ}$  associated to a polarization  $F$  is defined by the pairing of  $F$  with itself, so the BKS-kernel generalise the inner product.
- 2) One can use the pairing to extend the domain of the map  $O_{GQ}$  but on the extended domain  $O_{GQ}$  does in general no longer satisfy condition  $Q(iv)$ . The idea behind this extension is to use the flow  $\phi_t$  of the Hamiltonian vector field  $X_f$  on the phase space  $M$  to define a transformed polarization  $\phi_t^*F$ , then to construct the pairing between  $F$  and  $\phi_t^*F$  (if it exists!) and finally to take the derivative with respect to  $t$ . This method is sometimes called the method of infinitesimal pairing; it can be used for the kinetic energy  $|p|^2/2$  on the phase space  $\mathbb{R}^{2n}$  with the polarization  $F_r$  in which case it yields the correct answer  $O_{GQ}(|p|^2/2) = -\hbar^2 \Delta_r / 2$  on  $\mathcal{H}_r$ . The main drawback of this method is that there is no guarantee that the result is a self adjoint operator; one can not even tell beforehand whether it is formally symmetric; in each separate case one has to check whether the obtained operator is a self adjoint one.
- 3) Suppose we have two different polarizations  $F_1$  and  $F_2$  which define two Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , suppose that the BKS-kernel defines a unitary equivalence  $A: \mathcal{H}_1 \rightarrow \mathcal{H}_2$  between  $\mathcal{H}_1$  and  $\mathcal{H}_2$  and suppose that  $f$  is an observable which lies in the extended domains of both  $O_{GQ,1}$  and  $O_{GQ,2}$  as defined in the previous paragraph. One might hope that  $O_{GQ,1}(f)$  and  $O_{GQ,2}(f)$  are 'the same', i.e.  $A \circ O_{GQ,1}(f) = O_{GQ,2}(f) \circ A$ . Unfortunately, this is not true in general: the result of geometric quantization depends on the choice of the polarization, even when the Hilbert spaces are unitarily related (see [Tu2] for an example of this kind on  $M = \mathbb{R}^{2n}$ ). This 'negative' result is intimately related to the fact that geometric quantization tries to satisfy the incompatible conditions  $Q(i) \dots Q(v)'$  as far as possible (see [Ab&Ma] for some explicit calculations).

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# Space-time Algebra in the Dirac Theory

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## SPACE-TIME ALGEBRA STA

### 1.1. Introduction

Let  $M$  denote Minkowski space-time, that is  $\mathbb{R}^4$ , endowed with an orthonormal basis  $\{e_0, e_1, e_2, e_3\}$  and an indefinite quadratic form

$$(x, x) = \|x\|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2, \quad x \in M.$$

We want to define a multiplication of vectors in  $M$ , satisfying the rule  $x^2 = \|x\|^2$ . Using the basis  $\{e_0, e_1, e_2, e_3\}$  and writing

$$x = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3$$

this requirement can be expressed by

$$(x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3)^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

This holds for

$$\begin{cases} e_0^2 = 1 \\ e_1^2 = e_2^2 = e_3^2 = -1 \\ e_k e_l + e_l e_k = 0, \quad k \neq l \end{cases},$$

or shortly

$$e_k e_l + e_l e_k = 2g_{kl} = 2 \text{diag}(1, -1, -1, -1).$$

### REMARKS.

1. Clearly with the multiplication of vectors as introduced above we recovered the real Clifford algebra of  $M$ . This  $2^4$ -dimensional algebra is named *space-*

*time algebra STA.* As to be found in e.g. [1], volume 2, page 41, one can introduce STA in a coordinate free way by means of the tensor algebra of  $M$  and the ideal, generated by expressions of the form  $x \otimes x - (x, x) \cdot 1$ .

2. Do not confuse the 16-dimensional (real) Clifford algebra, just introduced, and the 32-dimensional complex Dirac algebra, as used in relativistic quantum mechanics and isomorphic with the algebra of complex  $(4 \times 4)$ -matrices  $\mathbb{C}(4)$ .

### 1.2. Matrix representations

Sometimes it is convenient to represent the basis vectors  $e_0, e_1, e_2, e_3$  by matrices. We mostly use the (unitary) representation  $\gamma_0, \gamma_1, \gamma_2, \gamma_3$  given by

$$\gamma_0 = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}, \quad \gamma_k = \begin{bmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{bmatrix}, \quad k = 1, 2, 3$$

where  $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $\sigma_k$  are the Pauli matrices given by

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Obviously one has

$$\gamma_k \gamma_0 = \begin{bmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{bmatrix}, \quad k = 1, 2, 3$$

$$\gamma_0 \gamma_1 \gamma_2 \gamma_3 = \gamma_5 = i \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}.$$

An alternative representation for STA is the algebra  $\mathbb{H}(2)$  of  $(2 \times 2)$ -matrices over the field of quaternions  $\mathbb{H}$ . In this case the basis vectors  $e_0, e_1, e_2, e_3$  respectively correspond to  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  and  $\begin{bmatrix} 0 & i_k \\ i_k & 0 \end{bmatrix}$ ,  $k = 1, 2, 3$ , where  $i_k$ ,  $k = 1, 2, 3$ , are the quaternions.

### 1.3. Some requisites in STA

Having made a choice for the orientation the *pseudoscalar*  $e_5 = e_0 e_1 e_2 e_3$  is an invariant, just like the determinant in linear spaces. Note that

$$e_5^2 = -1$$

and

$$e_5 e_k = -e_k e_5, \quad k = 0, 1, 2, 3.$$

Let  $a$  and  $b$  be vectors in STA, then we can write

$$ab = \frac{1}{2}(ab + ba) + \frac{1}{2}(ab - ba).$$

The *symmetric* part  $\frac{1}{2}(ab + ba) = a \cdot b$  is a scalar, because

$$ab + ba = (a + b)^2 - a^2 - b^2 \in \mathbb{R}.$$

The *antisymmetric* part  $\frac{1}{2}(ab - ba) = a \wedge b$  is called the bivectorial part of  $ab$ .

The elements of STA are called *multivectors* and can be written (with 16 parameters) as

$$A = \alpha + \alpha^k e_k + \alpha^{kl} e_k e_l + \alpha^{klm} e_k e_l e_m + \beta e_5,$$

$k < l < m$  and  $k, l, m = 0, 1, 2, 3$ .

We shall also write for the multivector  $A$ :

$$A = A_0 + A_1 + A_2 + A_3 + A_4$$

where

$A_0 = \alpha$  is the *scalar* part of  $A$ ,

$A_1 = \alpha^k e_k$  is the *vectorial* part of  $A$ ,

$A_2 = \alpha^{kl} e_k e_l$  is the *bivectorial* part of  $A$ ,

$A_3 = \alpha^{klm} e_k e_l e_m$  is the *trivectorial* part of  $A$  and

$A_4 = \beta e_5$  is the *pseudoscalar* part of  $A$ .

Next we introduce some special maps of  $\text{STA} \rightarrow \text{STA}$ .

The *inversion* or *main involution* is given by  $A \mapsto \bar{A}$  with

$$\bar{A} = -e_5 A e_5 = A_0 - A_1 + A_2 - A_3 + A_4.$$

It can also be given by  $\bar{A}_k = (-1)^k A_k$ . The most obvious properties are  $\bar{\bar{A}} = A$  and  $\overline{AB} = \bar{A}\bar{B}$ .

The *even* and *odd* parts of  $A$  are defined by

$$A_{\text{even}} = \frac{1}{2}(A + \bar{A}) = \frac{1}{2}(A - e_5 A e_5) = A_0 + A_2 + A_4$$

and

$$A_{\text{odd}} = \frac{1}{2}(A - \bar{A}) = \frac{1}{2}(A + e_5 A e_5) = A_1 + A_3.$$

The *reversion* or *main antiautomorphism* is given by  $A \mapsto \tilde{A}$  with

$$\tilde{A} = A_0 + A_1 - A_2 - A_3 + A_4.$$

It can also be given by

$$\tilde{A}_k = (-1)^{\frac{k(k-1)}{2}} A_k = (-1)^{\lfloor \frac{k}{2} \rfloor} A_k.$$

The most obvious properties are  $\tilde{\tilde{A}} = A$  and  $\tilde{\tilde{A}B} = \widetilde{BA}$ .

Note that  $A \rightarrow \tilde{A}$  reflects the order of the basis vectors  $e_0, e_1, e_2, e_3$ . Further we present the map  $A \mapsto A^\dagger$  by

$$A^\dagger = e_0 \tilde{A} e_0.$$

The most obvious properties of this map are  $(A^\dagger)^\dagger = A$  and  $A^\dagger B^\dagger = (BA)^\dagger$ . Observe that his map corresponds to Hermitean conjugation of matrices.

Finally we introduce the *Hodge dual map*  $A \mapsto *A$  by

$$*A = -\tilde{A}e_5.$$

REMARKS.

1. Besides the basis  $\{e_0, e_1, e_2, e_3\}$  we often use the *dual* basis  $\{e^0, e^1, e^2, e^3\}$  given by  $e^0 = e_0$  and  $e^k = -e_k$ ,  $k = 1, 2, 3$ .
2. STA is a special case of the Clifford algebra of  $\mathbb{R}^n$  with metric form  $(x, x) = x_1^2 + \dots + x_p^2 - (x_{p+1}^2 + \dots + x_n^2)$ . All these Clifford algebras are isomorphic to matrix algebras over  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{H}$  or over the direct products  ${}^2\mathbb{R}, {}^2\mathbb{C}, {}^2\mathbb{H}$ . All details can be found in [2] and in [3].

### 1.3. The Pauli algebra $P$

Let us introduce  $\epsilon_1, \epsilon_2, \epsilon_3$  by

$$\epsilon_k = e_k e_0, \quad k = 1, 2, 3$$

with properties

$$\epsilon_k \epsilon_l = e_l e_k, \quad k \neq l, \quad k, l = 1, 2, 3$$

and

$$\epsilon_1 \epsilon_2 \epsilon_3 = e_0 e_1 e_2 e_3 = e_5.$$

Obviously  $\epsilon_1, \epsilon_2$  and  $\epsilon_3$  generate the even subalgebra of STA. Because

$$\epsilon_k \epsilon_l + \epsilon_l \epsilon_k = 2\delta_{kl}, \quad k, l = 1, 2, 3$$

this even subalgebra is isomorphic to the Clifford algebra of  $\mathbb{R}^3$  with Euclidean metric, the so-called Pauli algebra  $P$ . Note that indeed the Pauli matrices  $\sigma_1, \sigma_2, \sigma_3$  are matrix representations of the basis vectors  $\epsilon_1, \epsilon_2, \epsilon_3$  and that the algebra  $\mathbb{C}(2)$  of complex  $(2 \times 2)$ -matrices is isomorphic to  $P$ .

More generally one can write down the chain of even subalgebras:

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset P \subset \text{STAC} \subset \dots$$

### 1.4. The differential operator $\partial$

Finally we introduce the differential operator  $\partial$  as

$$\partial = e_0 \partial_0 - e_1 \partial_1 - e_2 \partial_2 - e_3 \partial_3 = e^\mu \partial_\mu$$

with properties

$$\partial = d - \delta \quad (\text{Hodge-de Rham operator}),$$

$$\partial x = \partial \cdot x + \partial \wedge x,$$

$$\partial^2 = \partial \cdot \partial + \partial \wedge \partial = \partial \cdot \partial,$$

$$\partial^2 = -d\delta - \delta d \quad (\text{Laplace-Beltrami operator}).$$

Although one has the rule  $\partial_k(ab) = (\partial_k a)b + a(\partial_k b)$ , *Leibniz rule* for  $\partial$  of course cannot be expressed by  $\partial(ab) = (\partial a)b + a(\partial b)$ .

For more details and rules we refer to [4], Section 4.

## 2. DIRAC GAUGE THEORY IN STA

### 2.1. The Dirac equation

The Dirac equation, describing electrons and photons can be given by

$$(i\gamma^\mu D_\mu - m)\Psi = 0$$

with  $D_\mu = \partial_\mu - iqA_\mu$  and

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \alpha_1 + i\beta_1 \\ \alpha_2 + i\beta_2 \\ \alpha_3 + i\beta_3 \\ \alpha_4 + i\beta_4 \end{pmatrix} = (\alpha_1 + i\beta_1)u_1 + (\alpha_2 + i\beta_2)u_2 + \quad (1) \\ + (\alpha_3 + i\beta_3)u_3 + (\alpha_4 + i\beta_4)u_4$$

$$\text{with } u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, u_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ as basis vectors in spinor space.}$$

We can also write

$$(i\gamma^\mu \partial_\mu + q\gamma^\mu A_\mu)\Psi = m\Psi$$

or in slash-notation

$$\partial \Psi + qA \Psi = m\Psi \quad \text{D1}$$

with  $\partial = \gamma^\mu \partial_\mu$  and  $A = \gamma^\mu A_\mu$ .

### 2.2. The Dirac equation in STA

The main goal of this section is to give descriptions of Diracs equation in STA-language.

Using the obvious relations

$$\begin{aligned} \gamma_0 u_1 &= u_1, & u_2 &= \gamma_3 \gamma_1 u_1, \\ i u_1 &= \gamma_2 \gamma_1 u_1, & u_3 &= \gamma_3 \gamma_0 u_1, \\ & & u_4 &= \gamma_1 \gamma_0 u_1, \end{aligned}$$

one finds after substitution in (1)

$$\Psi = (\alpha_1 + \alpha_4 \gamma_1 \gamma_0 + \beta_4 \gamma_2 \gamma_0 + \alpha_3 \gamma_3 \gamma_0 + \beta_2 \gamma_3 \gamma_2 + \alpha_2 \gamma_3 \gamma_1 + \beta_1 \gamma_2 \gamma_1 + \beta_3 \gamma_5)u_1. \quad (2)$$

The quantity in parentheses appears exactly as the matrix representation of an even multivector  $\psi$  in STA. We write shortly  $\Psi = M(\psi)u_1$  and if no confusion

is likely we shall identify  $M(\psi)$  and  $\psi$ .

Substitution of  $\Psi = M(\psi)u_1$  in the Dirac equation

$$i\gamma^\mu \partial_\mu \Psi + q\gamma^\mu A_\mu \Psi = m\Psi$$

and identifying  $M(\psi)$  and  $\psi$  yields the equation

$$(\gamma^\mu \partial_\mu \psi \gamma_2 \gamma_1 \gamma_0 + q\gamma^\mu A_\mu \psi \gamma_0 - m\psi)u_1 = 0.$$

The expression in parentheses is the matrix representation of an *even* multivector in STA and although  $u_1$  does not have an inverse, we can write

$$\gamma^\mu \partial_\mu \psi \gamma_2 \gamma_1 \gamma_0 + q\gamma^\mu A_\mu \psi \gamma_0 - m\psi = 0$$

because

$$M(\psi) = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \\ \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \end{pmatrix}$$

is determined by its first column  $M(\psi)u_1$ .

Changing from matrices to multivectors in STA we find for Dirac's equation

$$\partial\psi e_2 e_1 + qA\psi = m\psi e_0$$

or, using  $e_2 e_1 = e_5 e_3 e_0$

$$\partial\psi e_5 e_3 e_0 + qA\psi = m\psi e_0, \quad \text{D2}$$

with  $\psi$  an *even* multivector in STA.

Obviously, if one starts with D2, one finds D1 and thence the equivalence of D1 and D2 is clear.

#### REMARKS.

1. Note that the scalar part of  $\psi$  corresponds to  $\frac{1}{4}\text{tr} M(\psi)$  because all  $\gamma_k \gamma_l$  with  $k \neq l$  and  $\gamma_5$  are traceless.
2. In contradistinction to  $\gamma_0, \gamma_1$  and  $\gamma_2$  the quantities  $e_0, e_1$  and  $e_2$  in D2 are no matrices in  $\mathbb{C}(4)$  but vectors in STA.
3. The quantity  $i \in \mathbb{C}$  in D1 has been replaced by  $e_5 e_3 e_0 = e_2 e_1$  in D2. Indeed one has  $(e_2 e_1)^2 = -1$  but the bivector  $e_2 e_1$  represents some special direction in STA. The interpretation of this direction has been amply discussed in [5].
4. As known, other representations of the  $\gamma$ -matrices are related to our choice by the relation  $\hat{\gamma}_\mu = S^{-1} \gamma_\mu S$ . One can prove that apart from an (allowed) Lorentz transformation, equation D2 is independent of the choice of the  $\gamma_\mu$ . All details are discussed by Hestenes in [5].



### 2.3. Another description of Dirac's equation in STA

Multiplication from the right of equation D2 with the factor  $\frac{1}{2}(1+e_0)(1+e_3e_0)$  and introduction of the multivector

$$\phi = \frac{1}{2}\psi(1+e_0)(1+e_3e_0)$$

yields after some minor manipulations:

$$\partial\phi e_5 + qA\phi = m\phi \quad \text{D3}$$

with the condition  $\phi e_3 e_0 = \phi$ . Conversely the equation D3 yields again D2 as can be proved in the following way:

Substitution of  $\phi = \psi_1 + \psi_2 e_0$  with  $\psi_1$  and  $\psi_2$  even in D3 and use of

$$\partial\phi e_5 = \partial\phi e_3 e_0 e_5 = \partial\phi e_2 e_1$$

gives the equation

$$\partial\psi_1 e_2 e_1 + \partial\psi_2 e_1 e_0 + qA\psi_1 + qA\psi_2 e_0 = m\psi_1 + m\psi_2 e_0.$$

After splitting in odd and even parts one finds

$$\partial\psi_1 e_2 e_1 + qA\psi_1 = m\psi_2 e_0$$

and

$$\partial\psi_2 e_0 e_2 e_1 + qA\psi_2 e_0 = m\psi_1.$$

Multiplication of the second equation on the right by  $e_0$  gives the pair of equations

$$\partial\psi_1 e_5 e_3 e_0 + qA\psi_1 = m\psi_2 e_0$$

$$\partial\psi_2 e_5 e_3 e_0 + qA\psi_2 = m\psi_1 e_0.$$

Add both equations and call  $\psi_1 + \psi_2 = \psi$  ( $\psi$  even), thence one finds

$$\partial\psi e_5 e_3 e_0 + qA\psi = m\psi e_0. \quad \text{D2}$$

### 2.4. Remarks

1. It is tempting to conclude that D1 and D3 have more resemblance than D1 and D2 but that is misleading because the anomaly in D3 is hidden in the structure of  $\phi = \frac{1}{2}\psi(1+e_0)(1+e_3e_0)$ . We return to that question in the next section.
2. Although the equivalence of D1 and D3 is evident from that of D1 and D2 and of D2 and D3 we give a straight proof because it affords more insight in the structure of D3.

Let

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \text{ and } \eta = \begin{pmatrix} \psi_1 & 0 & \psi_1 & 0 \\ \psi_2 & 0 & \psi_2 & 0 \\ \psi_3 & 0 & \psi_3 & 0 \\ \psi_4 & 0 & \psi_4 & 0 \end{pmatrix}.$$

As is easy to check one has the relations  $\eta\gamma_3\gamma_0 = \eta$  and  $\eta\gamma_5 = \eta i$ . Conversely, if  $\alpha \in \mathbb{C}(4)$  and the relations  $\alpha\gamma_3\gamma_0 = \alpha$  and  $\alpha\gamma_5 = \alpha i$  hold, then

$$\alpha = \begin{pmatrix} \psi_1 & 0 & \psi_1 & 0 \\ \psi_2 & 0 & \psi_2 & 0 \\ \psi_3 & 0 & \psi_3 & 0 \\ \psi_4 & 0 & \psi_4 & 0 \end{pmatrix}.$$

Dirac's Equation D1, given by

$$i\gamma^\mu D_\mu \Psi = m\Psi \quad \text{with} \quad D_\mu = \partial_\mu - iqA_\mu$$

is equivalent with the equation

$$i\gamma^\mu D_\mu \eta = m\eta$$

or

$$\gamma^\mu D_\mu \eta \gamma_5 = m\eta.$$

Because  $\eta \in \mathbb{C}(4)$ , one can write  $\eta = \phi + i\phi_1$ ;  $\phi, \phi_1 \in \text{STA}$ . By using  $\eta i = \eta\gamma_5$  one finds  $\phi_1 = -\phi\gamma_5$ .

Substitution of  $\eta = \phi + i\phi_1$  in  $\gamma^\mu D_\mu \eta \gamma_5 = m\eta$  and use of  $\phi_1 = -\phi\gamma_5$  yields the pair of equations

$$\gamma^\mu D_\mu \phi \gamma_5 = m\phi \quad \text{and} \quad \gamma^\mu D_\mu \phi_1 \gamma_5 = m\phi_1$$

both equivalent with the sole equation

$$\gamma^\mu D_\mu \phi \gamma_5 = m\phi,$$

with the condition  $\phi\gamma_3\gamma_0 = \phi$  and with  $D_\mu \phi = \partial_\mu \phi - qA_\mu \phi \gamma_5$ . Summarizing one recognizes the matrix representation of D3. The converse is also true.

3. The equation of Dirac is related to a number of notions and quantities such as currents, gauge invariant derivative, Lagrange density and so on. In the next table we summarize these associated notions for the Equations D1, D2, D3.

	D1	D2	D3
Dirac equation	$\partial \Psi_i + qA \Psi = m \Psi$ $\Psi \in \mathbb{C}^4$	$\partial \psi e_5 e_3 e_0 + qA \psi = m \psi e_0$ $\psi \in \text{STA}$ $\psi$ even	$\partial \phi e_5 + qA \phi = m \phi$ $\phi \in \text{STA}$ $\phi e_3 e_0 = \phi$
current	$J_\mu = \Psi^\dagger \gamma_0 \gamma_\mu \Psi$	$J = \psi e_0 \bar{\psi}$	$J_\mu = (\phi^\dagger e_0 e_\mu \phi)_0$
spin current	$s_\mu = -i \Psi^\dagger \gamma_0 \gamma_\mu \gamma_5 \Psi$	$s = \psi e_3 \bar{\psi}$	$s_\mu = -(\phi^\dagger e_0 e_\mu e_5 \phi e_5)_0$
gauge transformation	$\hat{\Psi} = \Psi e^{-i\alpha}$	$\hat{\psi} = \psi e^{-ae_5 e_3 e_0}$	$\hat{\phi} = \phi e^{-ae_5}$
gauge invariant derivative	$D_\mu \Psi = \partial_\mu \Psi - qA \Psi_i$	$D\psi = \partial\psi - qA \psi e_5 e_3 e_0$	$D\phi = \partial\phi - qA \phi e_5$
Lagrangian density	$L = \Psi^\dagger \gamma_0 (\gamma^\mu D_\mu - m) \Psi$	$L = (\psi^\dagger e_0 (D\psi e_5 e_3 e_0 - m\psi e_0))_0$	$L = (\phi^\dagger e_0 (D\phi e_5 - m\phi))_0$

All derivations can be found in [4].

### 3. DIRAC THEORY FOR PAIRS OF PARTICLES

#### 3.1. Classical theory

The strong forces between nucleons and the weak forces between leptons are described, at least in former days, by the pair of Dirac equations:

$$(i\gamma^\mu D_\mu - m)\Psi = 0$$

with  $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$ ,  $D_\mu = \partial_\mu - \frac{1}{2}iA_\mu^k \tau_k$  and  $\tau_k = \sigma_k$ ,  $k = 1, 2, 3$ .

We shall use a description equivalent with this classical one, although, with the SU(2)-action from the right side on  $\Psi$ .

Let

$$\Psi_1 = \begin{pmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{31} \\ \psi_{41} \end{pmatrix}, \quad \Psi_2 = \begin{pmatrix} \psi_{12} \\ \psi_{22} \\ \psi_{32} \\ \psi_{42} \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \\ \psi_{31} & \psi_{32} \\ \psi_{41} & \psi_{42} \end{pmatrix},$$

then the free Dirac equation for the pair  $(\Psi_1, \Psi_2) = \Psi$  can be written as

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0.$$

Replacing of  $\partial_\mu \Psi$  by  $D_\mu \Psi = \partial_\mu \Psi + \Psi A_\mu$  with  $A_\mu = \frac{1}{2}iA_\mu^k \tau_k$  yields

$$(i\gamma^\mu D_\mu - m)\Psi = 0.$$

In the next table we compare some notions, associated with the Dirac

equation, in the classical description and in our one.

	classical	here
current	$j_k^\mu = \frac{1}{2} \bar{\Psi} \gamma^\mu \tau_k \Psi$	$j_k^\mu = \frac{1}{2} \text{tr}(\bar{\Psi} \gamma^\mu \Psi \tau_k)$
gauge transformations	$\hat{\psi} = U \psi, U = e^{-\frac{1}{2} i \alpha^k e_k}$ $A_\mu = U A_\mu U^{-1} - (\partial_\mu U) U^{-1}$	$\hat{\psi} = \psi U, U = e^{-\frac{1}{2} i \alpha^k \tau_k}$ $A_\mu = U^{-1} A_\mu U - U^{-1} (\partial_\mu U)$
gauge invariant derivative	$D_\mu = \partial_\mu + A_\mu$	$D_\mu \Psi = \partial_\mu \Psi + \Psi A_\mu$
Lagrangian	$L_1 = \bar{\Psi} (i \gamma^\mu D_\mu - m) \Psi$	$L_1 = \text{tr}(\bar{\Psi} (i \gamma^\mu D_\mu - m) \Psi)$
Field strength tensor	$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$	$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$

The rule  $[D_\mu, D_\nu] \Psi = F_{\mu\nu} \Psi$  transforms into  $[D_\mu, D_\nu] \Psi = \Psi F_{\mu\nu}$ .

### 3.2. $(4 \times 4)$ -matrices as wave functions

As a prelude to a description of 3.1 in STA-formalism we need the following conventions, again resulting in the classical theory. Instead of

$$\Psi_1 = \begin{pmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{31} \\ \psi_{41} \end{pmatrix}, \quad \Psi_2 = \begin{pmatrix} \psi_{12} \\ \psi_{22} \\ \psi_{32} \\ \psi_{42} \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \\ \psi_{31} & \psi_{32} \\ \psi_{41} & \psi_{42} \end{pmatrix}$$

we consider the  $(4 \times 4)$ -matrices

$$\Phi_1 = \begin{pmatrix} \psi_{11} & 0 & \psi_{11} & 0 \\ \psi_{21} & 0 & \psi_{21} & 0 \\ \psi_{31} & 0 & \psi_{31} & 0 \\ \psi_{41} & 0 & \psi_{41} & 0 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} 0 & \psi_{12} & 0 & \psi_{12} \\ 0 & \psi_{22} & 0 & \psi_{22} \\ 0 & \psi_{32} & 0 & \psi_{32} \\ 0 & \psi_{42} & 0 & \psi_{42} \end{pmatrix}$$

and

$$\Phi = \Phi_1 + \Phi_2 = \begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} & \psi_{21} & \psi_{22} \\ \psi_{31} & \psi_{32} & \psi_{31} & \psi_{32} \\ \psi_{41} & \psi_{42} & \psi_{41} & \psi_{42} \end{pmatrix}.$$

The pair of free Dirac equations now can be presented by

$$(i \gamma^\mu \partial_\mu - m) \Phi = 0. \quad (3)$$

As is easy to check,  $\Phi$  is an eigenfunction of  $\gamma_5 = i \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$  from the right with eigenvalue  $i \in \mathbb{C}$ , i.e.

$$i\Phi = \Phi i = \Phi \gamma_5.$$

Thence we write instead of (3):

$$\gamma^\mu \partial_\mu \Phi \gamma_5 = m\Phi.$$

Instead of  $A_\mu = -\frac{1}{2}iA_\mu^k \tau_k$  we now find  $A_\mu = -\frac{1}{2}A_\mu^k \gamma_k \gamma_0 \gamma_5$ . (Note that  $\gamma_k \gamma_0 = \tau_k \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$ .)

Finally all these conventions lead to the Dirac equation

$$\gamma^\mu D_\mu \Phi \gamma_5 = m\Phi, \quad \text{or}$$

$$\mathcal{D} \Phi \gamma_5 = m\Phi$$

where  $\mathcal{D} = \gamma^\mu D_\mu$ .

Our wave function  $\Phi$  is a  $4 \times 4$ -matrix over the complex field  $\mathbb{C}$  and hence belongs to the complex Dirac algebra, which is isomorphic to  $\mathbb{C}(4)$ .

Therefore we can write

$$\Phi = \alpha + \alpha^k \gamma_k + \alpha^{kl} \gamma_k \gamma_l + \alpha^{klm} \gamma_k \gamma_l \gamma_m + \beta \gamma_5,$$

where  $\alpha, \alpha^k, \alpha^{kl}, \alpha^{klm}$  and  $\beta \in \mathbb{C}$ .

Let  $\phi = \text{re} \Phi$  be the real part of  $\Phi$  then  $\phi$  is the matrix representation of an element of STA.

Obviously  $\mathcal{D} \Phi \gamma_5 = m\Phi$  implies  $\mathcal{D} \phi \gamma_5 = m\phi$  but the converse is also true. Viz. let  $\Phi = \phi + i\eta$ , then using  $\Phi i = \Phi \gamma_5$  one finds  $\eta = -\phi \gamma_5$ , i.e.  $\Phi = \phi(1 - i\gamma_5)$ . Therefore it obviously follows that  $\mathcal{D} \phi \gamma_5 = m\phi$  implies  $\mathcal{D} \Phi \gamma_5 = m\Phi$ .

Now we are ready to translate the equation  $\mathcal{D} \phi \gamma_5 = m\phi$  into the STA-formalism.

### 3.3. Dirac's equation in STA for pairs of particles

Meanwhile we have reasons enough to investigate the equation

$$D\phi e_5 = m\phi$$

with  $\phi \in \text{STA}$ ,  $D = e^\mu D_\mu$ ,  $D_\mu \phi = \partial_\mu \phi + \phi A_\mu$ ,  $A_\mu = -\frac{1}{2}A_\mu^k e_k e_0 e_5$  and with matrix representation  $\mathcal{D} \phi \gamma_5 = m\phi$ . For the associated notions one finds respectively

a) currents:

$$j_k^\mu = (\phi^\dagger e_0 e^\mu \phi e_k e_0)_0$$

b) gauge transformations:

$$\begin{cases} \hat{\phi} = \phi U \text{ with } U = e^{-\frac{1}{2}\alpha^k e_k e_0 e_3} \\ \hat{A}_\mu = U^{-1} A_\mu U - U^{-1}(\partial_\mu U) \end{cases}$$

c) Lagrangian:

$$L = 2(\phi^\dagger e_0(D\phi e_3 - m\phi))_0 - \frac{1}{4}F_{\mu\nu}^k F_k^{\mu\nu}$$

It is merely an algebraic exercise to prove the following gauge invariances.

1.  $(\hat{\phi}^\dagger e_0 e_\mu \hat{\phi})_0 = (\phi^\dagger e_0 e_\mu \phi)_0$
2.  $\hat{D}_\mu \hat{\phi} = (D_\mu \phi)U$
3.  $\hat{F}_{\mu\nu} = U^{-1} F_{\mu\nu} U$
4.  $\hat{L} = L$ .

### 3.4. Ideals in STA

Ideals in an algebra are generated by idempotents, minimal ideals by primitive idempotents. As is clear from the  $\mathbb{H}(2)$ -representation STA has two minimal left ideals  $J_+$  and  $J_-$ . (Compare again the complex Dirac algebra  $\cong \mathbb{C}(4)$  with four minimal left ideals.)

For the primitive idempotents we choose  $\frac{1}{2}(1 + e_3 e_0)$  and  $\frac{1}{2}(1 - e_3 e_0)$ . Hence we can write

$$\phi = \frac{1}{2}\phi(1 + e_3 e_0) + \frac{1}{2}\phi(1 - e_3 e_0)$$

with

$$\frac{1}{2}\phi(1 + e_3 e_0) = \phi_1 \in J_+$$

and

$$\frac{1}{2}\phi(1 - e_3 e_0) = \phi_2 \in J_-.$$

Observe that  $\phi_1 = \phi_1 e_3 e_0$  and thence we recognize the uninterpreted condition associated with D3 in 2.3. As promised in 2.4 we here return to that question by mentioning that D3 describes the restriction of  $D\phi e_3 = m\phi$  to the ideal  $J_+$  in STA.

This gives rise to the idea to describe the weak interaction between electron and neutrino by means of the two ideals  $J_+$  and  $J_-$ . For a further investigation we need the following trivial properties:

- a.  $J_+ \cap J_- = \{0\}$
- b.  $\phi_1 \in J_+$  iff  $\phi_1 e_3 e_0 = \phi_1$   
iff  $\phi_1(1 - e_3 e_0) = 0$  iff  $\phi_1(1 + e_3 e_0) = 2\phi_1$

and

$$\begin{aligned} \text{c. } \phi_2 \in J_- \text{ iff } \phi_2 e_3 e_0 = -\phi_2 \\ \text{iff } \phi_2(1 + e_3 e_0) = 0 \text{ iff } \phi_2(1 - e_3 e_0) = 2\phi_2. \end{aligned}$$

Substitution of  $\phi = \phi_1 + \phi_2$  in the equation  $e^\mu D_\mu \phi e_5 = m\phi$  yields:

$$\begin{aligned} 0 &= e^\mu D_\mu \phi e_5 - m\phi = \partial \phi e_5 + e^\mu \phi A_\mu e_5 - m\phi = \\ &= \partial \phi_1 e_5 + e^\mu \phi_1 A_\mu e_5 - m\phi_1 + \partial \phi_2 e_5 + e^\mu \phi_2 A_\mu e_5 - m\phi_2, \end{aligned}$$

where

$$\phi_a A_\mu e_5 = \frac{1}{2} A_\mu^k \phi_a e_k e_0 \quad a=1,2.$$

Some rewriting yields:

$$0 = (\partial \phi_1 e_5 + e^\mu B_{1,\mu} - m\phi_1) + (\partial \phi_2 e_5 + e^\mu B_{2,\mu} - m\phi_2)$$

where

$$B_{1,\mu} = \frac{1}{2} A_\mu^1 \phi_2 e_1 e_0 + \frac{1}{2} A_\mu^2 \phi_2 e_2 e_0 + \frac{1}{2} A_\mu^3 \phi_1 e_3 e_0 \in J_+$$

and

$$B_{2,\mu} = \frac{1}{2} A_\mu^1 \phi_1 e_1 e_0 + \frac{1}{2} A_\mu^2 \phi_1 e_2 e_0 + \frac{1}{2} A_\mu^3 \phi_2 e_3 e_0 \in J_-.$$

Whence

$$\partial \phi_1 e_5 + e^\mu B_{1,\mu} - m\phi_1 \in J_+$$

and

$$\partial \phi_2 e_5 + e^\mu B_{2,\mu} - m\phi_2 \in J_-$$

and therefore

$$\partial \phi_1 e_5 + e^\mu B_{1,\mu} - m\phi_1 = 0$$

and

$$\partial \phi_2 e_5 + e^\mu B_{2,\mu} - m\phi_2 = 0.$$

Substitution of the expressions for  $B_{1,\mu}$  and  $B_{2,\mu}$  and again some rewriting yields in  $J_+$ :

$$e^\mu (\partial_\mu \phi_1 e_5 + \frac{1}{2} A_\mu^3 \phi_1) + \frac{1}{2} e^\mu (A_\mu^1 \phi_2 + A_\mu^2 \phi_2 e_5) e_1 e_0 = m\phi_1 \quad (\alpha)$$

and similarly in  $J_-$ :

$$e^\mu (\partial_\mu \phi_2 e_5 - \frac{1}{2} A_\mu^3 \phi_2) + \frac{1}{2} e^\mu (A_\mu^1 \phi_1 - A_\mu^2 \phi_1 e_5) e_1 e_0 = m\phi_2. \quad (\beta)$$

Let us now compare  $(\alpha)$  and  $(\beta)$  with the description given in 3.1.

We started from the equation

$$(i\gamma^\mu D_\mu - m)\Psi = 0$$

with

$$D_\mu \Psi = \partial_\mu \Psi + \Psi A_\mu$$

$$A_\mu = -\frac{1}{2} i A_\mu^k \tau_k$$

and

$$\Psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \\ \psi_{31} & \psi_{32} \\ \psi_{41} & \psi_{42} \end{pmatrix} = (\Psi_1, \Psi_2).$$

Substitution of  $\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ ,  $\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ , and some rewriting and splitting yields now:

$$\gamma^\mu (\partial_\mu \Psi_1 i + \frac{1}{2} A_\mu^3 \Psi_1) + \frac{1}{2} \gamma^\mu (A_\mu^1 \Psi_2 + A_\mu^2 \Psi_2 i) = m \Psi_1 \quad (\alpha')$$

$$\gamma^\mu (\partial_\mu \Psi_2 i - \frac{1}{2} A_\mu^3 \Psi_2) + \frac{1}{2} \gamma^\mu (A_\mu^1 \Psi_1 - A_\mu^2 \Psi_1 i) = m \Psi_2. \quad (\beta')$$

Note that the only genuine difference between  $\alpha, \beta$  and  $\alpha', \beta'$  is the factor  $e_1 e_0$  in  $\alpha, \beta$  but that is just the factor that maps  $\phi_2 \in J_-$  onto  $\phi_2 e_1 e_0 \in J_+$  and similarly  $\phi_1 \in J_+$  onto  $\phi_1 e_1 e_0 \in J_-$ .

Hence our conclusion is that the description of the weak interaction with the two minimal ideals in STA is wholly equivalent with the usual description by means of the Lie group SU(2).

#### 4. STRONG INTERACTIONS AND STA

##### 4.1. Classical descriptions

Nowadays descriptions of strong interaction fields make use of the symmetry group SU(3) instead of SU(2). In this subsection we give a brief summary of SU(3) gauge fields as can be found in [6] but with the same adjustment as carried out for SU(2) in 3.1. (It turns out that the operator  $i\gamma^\mu \partial_\mu - m$  acts from the left but that the group SU(3) acts from the right.)

Let us first write down the free Dirac equations for triples of quarks:

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0$$

where

$$\Psi = (\Psi_{\text{red}}, \Psi_{\text{blue}}, \Psi_{\text{green}}) = \begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{31} & \psi_{32} & \psi_{33} \\ \psi_{41} & \psi_{42} & \psi_{43} \end{pmatrix}.$$



Subsequently we replace the operator  $\partial_\mu$  by  $D_\mu$ , defined as

$$D_\mu \Psi = \partial_\mu \Psi + \Psi A_\mu$$

where

$$A_\mu = A_\mu^k T_k = -\frac{1}{2} i A_\mu^k \lambda_k.$$

The  $(3 \times 3)$ -matrices  $\lambda_k$  are the well-known generators of  $SU(3)$ . The gauge transformations are given by

$$\begin{cases} \hat{\Psi} = \Psi U \\ \hat{A}_\mu = U^{-1} A_\mu U - U^{-1} \partial_\mu U \end{cases} \quad \text{with} \quad U = e^{-\frac{1}{2} i \alpha^k \lambda_k}.$$

The currents  $\lambda_k^\mu$  are in this case

$$\gamma_k^\mu = \frac{1}{2} \text{tr} \bar{\Psi} \gamma^\mu \Psi \lambda_k.$$

The wave function<sup>1</sup>  $\Psi = (\Psi_1, \Psi_2, \Psi_3)$  is associated with the Lagrangian

$$L_0 = \sum_{\alpha=1}^3 \bar{\Psi}_\alpha (i \gamma^\mu \partial_\mu - m) \Psi_\alpha = \text{tr}(\bar{\Psi} (i \gamma^\mu \partial_\mu - m) \Psi).$$

As in the  $SU(2)$ -case replacing of  $\partial_\mu \Psi$  by  $D_\mu \Psi = \partial_\mu \Psi + \Psi A_\mu$  yields

$$L_1 = \text{tr}(\bar{\Psi} (i \gamma^\mu D_\mu - m) \Psi).$$

For the dynamical part  $L_F$  of  $L$  one has

$$L_F = -\frac{1}{2} \text{tr}(F_{\mu\nu} F^{\mu\nu}) = -\frac{1}{2} F_{\mu\nu}^k F_{\mu\nu}^k$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] = -\frac{1}{2} i F_{\mu\nu}^k \lambda_k.$$

Finally we summarize the usual gauge invariances:

1.  $\hat{D}_\mu \hat{\Psi} = (D_\mu \Psi) U$
2.  $\hat{F}_{\mu\nu} = U^{-1} F_{\mu\nu} U$
3.  $\hat{L} = L.$

The proofs can be found in[6].

#### 4.2. $SU(3)$ gauge theory in STA

In contradistinction to  $SU(2) \subset STA$  the group  $SU(3)$  suffers from the disease that it is not a subset of STA. Therefore a treatment as in 3.2 is now impossible.

<sup>1</sup> From now on we drop the folkloristic notation  $\Psi_{\text{red}}$  and so on.

In spite of this defect it is possible to describe the notions and properties of Section 4.1 in terms of STA. Consider to that end the Lie algebra of bivectors in STA. We can write any bivector  $B$  as

$$B = (a_1 + e_5 b_1)e_1 e_0 + (a_2 + e_5 b_2)e_2 e_0 + (a_3 + e_5 b_3)e_3 e_0.$$

Note that one can consider this space of bivectors as a copy of  $\mathbb{C}^3$ . Next observe that

$$B^\dagger = (a_1 - e_5 b_1)e_1 e_0 + (a_2 - e_5 b_2)e_2 e_0 + (a_3 - e_5 b_3)e_3 e_0$$

can be considered as the ‘complex conjugate’ of  $B$ .

Further, we find that

$$B^\dagger B = a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2 - 2 \begin{vmatrix} e_1 e_0 & e_2 e_0 & e_3 e_0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Therefore  $(B^\dagger B)_0 = a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2$  and that means that the Lie algebra of bivectors in STA with square norm  $\|B\|^2 = (B^\dagger B)_0$  has  $SU(3)$  as invariance group.

The original ideas can be found in [7]. To develop further details it is necessary to express the generators  $\lambda_k$  in terms of STA. The procedure is as follows:

Let  $B$  be a bivector in STA, define  $B\lambda_1 = \frac{1}{2}(e_1 e_0 B e_2 e_0 + e_2 e_0 B e_1 e_0)$ . Substitution of

$$B = (a_1 + e_5 b_1)e_1 e_0 + (a_2 + e_5 b_2)e_2 e_0 + (a_3 + e_5 b_3)e_3 e_0$$

yields

$$B\lambda_1 = (a_2 + e_5 b_2)e_1 e_0 + (a_1 + e_5 b_1)e_2 e_0.$$

On basis  $\{e_1 e_0, e_2 e_0, e_3 e_0\}$  one finds indeed  $\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ . Likewise we can calculate the remaining  $\lambda_k$ :

$$B\lambda_2 = \frac{1}{2}(e_3 e_0 B - B e_3 e_0) \quad \lambda_2 = \begin{pmatrix} 0 & -e_5 & 0 \\ e_5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where  $e_5$  plays the role of  $i$ .

$$B\lambda_3 = \frac{1}{2}(e_1 e_0 B e_1 e_0 - e_2 e_0 B e_2 e_0) \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B\lambda_4 = \frac{1}{2}(e_1 e_0 B e_3 e_0 + e_3 e_0 B e_1 e_0) \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$B\lambda_5 = \frac{1}{2}(e_2e_0B - Be_2e_0) \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -e_5 \\ 0 & 0 & 0 \\ e_5 & 0 & 0 \end{pmatrix}$$

$$B\lambda_6 = \frac{1}{2}(e_2e_0Be_3e_0 + e_3e_0Be_2e_0) \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$B\lambda_7 = \frac{1}{2}(e_1e_0B - Be_1e_0) \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -e_5 \\ 0 & e_5 & 0 \end{pmatrix}$$

$$B\lambda_8 = \frac{-1}{2\sqrt{3}}(B + 3e_3e_0Be_3e_0) \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Given  $\Psi = (\Psi_1, \Psi_2, \Psi_3)$

$$\text{with } \Psi_1 = \begin{pmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{31} \\ \psi_{41} \end{pmatrix} = \begin{pmatrix} a_{11} + ib_{11} \\ a_{21} + ib_{21} \\ a_{31} + ib_{31} \\ a_{41} + ib_{41} \end{pmatrix}$$

and likewise  $\Psi_{kl} = a_{kl} + ib_{kl}$ ,  $1 \leq k \leq 4$ ,  $1 \leq l \leq 3$  one can consider the  $4 \times 3$ -matrix

$$\psi_{kl} = (a + e_5 b)_{kl}.$$

The only difference between  $\Psi$  and  $\psi$  is that the complex number  $i$  has been replaced by the pseudo-unit  $e_5$ . Introducing the linear space  $B^4$ , consisting of quadruples of bivectors i.e.

$$\phi = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} \in B^4$$

we can write

$$\begin{aligned}\phi &= \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} = \psi \begin{pmatrix} e_1 e_0 \\ e_2 e_0 \\ e_3 e_0 \end{pmatrix} = \\ &= \begin{pmatrix} a_{11}e_1e_0 + b_{11}e_5e_1e_0 + a_{12}e_2e_0 + b_{12}e_5e_2e_0 + a_{13}e_3e_0 + b_{13}e_5e_3e_0 \\ a_{21}e_1e_0 + b_{21}e_5e_1e_0 + a_{22}e_2e_0 + b_{22}e_5e_2e_0 + a_{23}e_3e_0 + b_{23}e_5e_3e_0 \\ a_{31}e_1e_0 + b_{31}e_5e_1e_0 + a_{32}e_2e_0 + b_{32}e_5e_2e_0 + a_{33}e_3e_0 + b_{33}e_5e_3e_0 \\ a_{41}e_1e_0 + b_{41}e_5e_1e_0 + a_{42}e_2e_0 + b_{42}e_5e_2e_0 + a_{43}e_3e_0 + b_{43}e_5e_3e_0 \end{pmatrix}.\end{aligned}$$

We yet define the  $4 \times 4$ -matrices  $\Gamma^0, \Gamma^1, \Gamma^2, \Gamma^3$  as  $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ , but where  $i \in \mathbb{C}$  (in  $\gamma^2$ ) is replaced by  $e_5$ .

The free Dirac equations

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0,$$

as mentioned in 4.1 can be replaced by

$$(e_5 \Gamma^\mu \partial_\mu - m)\psi = 0.$$

Obviously this yields the equations

$$(e_5 \Gamma^\mu \partial_\mu - m)\phi = 0.$$

Just like in Section 3 we can replace  $\partial_\mu \phi$  by

$$D_\mu \phi = \partial_\mu \phi + \phi A_\mu,$$

to be interpreted in the following way

$$\phi \lambda_k = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} \lambda_k = \begin{pmatrix} B_1 \lambda_k \\ B_2 \lambda_k \\ B_3 \lambda_k \\ B_4 \lambda_k \end{pmatrix},$$

the latter as defined above and

$$\phi A_\mu = A_\mu^k \phi T_k = -\frac{1}{2} e_5 A_\mu^k \phi \lambda_k$$

whence

$$D_\mu \phi = \partial_\mu \phi + \phi A_\mu.$$

In conclusion we find

$$(e_5 \Gamma^\mu D_\mu - m)\phi = 0$$

or

$$D\phi e_5 = m\phi$$

with

$$\phi \in B^4.$$

REMARK. Comparing the equation  $D\phi e_5 = m\phi$  with  $\phi \in STA$  as dealt with in Section 3 one observes that in that case  $D\phi e_5 = m\phi$  obeys Lorentz invariance because it is manifestly independent of coordinates, but for the equation  $D\phi e_5 = m\phi$  with  $\phi \in B^4$  as presented in this section this Lorentz invariance is not automatically satisfied. This Lorentz invariance can be proved in a similar way as done in the literature on equation D1.

Compare for instance [8], page 52.

Finally we give the translations into STA of the notions  $F_{\mu\nu}, \hat{\phi}, L_0, L_1, L_F$  and so on, including their invariances. Defining

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$$

one finds the property

$$[D_\mu, D_\nu]\phi = \phi F_{\mu\nu}.$$

PROOF.

$$\begin{aligned} [D_\mu, D_\nu]B_k &= D_\mu(D_\nu B_k) - D_\nu(D_\mu B_k) = \\ &= D_\mu(\partial_\nu B_k + B_k A_\mu) - D_\nu(\partial_\mu B_k + B_k A_\mu) = \\ &= \partial_\mu(\partial_\nu B_k + B_k A_\nu) + (\partial_\nu B_k + B_k A_\nu)A_\mu + \\ &\quad - \partial_\nu(\partial_\mu B_k + B_k A_\mu) - (\partial_\mu B_k + B_k A_\mu)A_\nu = \\ &= B_k(\partial_\mu A_\nu - \partial_\nu A_\mu + A_\nu A_\mu - A_\mu A_\nu) = B_k F_{\mu\nu}, \end{aligned}$$

hence we can write

$$[D_\mu, D_\nu]\phi = \phi F_{\mu\nu}$$

where

$$\phi = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}.$$

□

Next define  $\phi U$  as

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} U = \begin{bmatrix} B_1 U \\ B_2 U \\ B_3 U \\ B_4 U \end{bmatrix}$$

and thence

$$\begin{cases} \hat{\phi} = \phi U \\ \hat{A}_\mu = U^{-1} A_\mu U - U^{-1} (\partial_\mu U) \end{cases}$$

This expresses the gauge transformation for SU(3) in STA-formalism.

Let  $\phi^\dagger = (B_1^\dagger, B_2^\dagger, B_3^\dagger, B_4^\dagger)$  then we can define the currents by its components

$$j_k^\mu = \frac{1}{2} (\phi^\dagger \Gamma_0 \Gamma^k \phi \lambda_k)_0.$$

This expression corresponds to

$$j_k^\mu = \frac{1}{2} \text{tr}(\Psi^\dagger \gamma_0 \gamma^\mu \Psi \lambda_k)$$

where

$$\Psi = (\psi_1, \psi_2, \psi_3).$$

which we prove as follows:

$$\phi^\dagger \Gamma_0 \Gamma^k \phi \lambda_k = (e_1 e_0, e_2 e_0, e_3 e_0) \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{\psi}_3 \end{pmatrix} \Gamma_0 \Gamma^\mu (\psi_1, \psi_2, \psi_3) \begin{pmatrix} e_1 e_0 \\ e_2 e_0 \\ e_3 e_0 \end{pmatrix} \lambda_k.$$

Hence

$$\frac{1}{2} (\phi^\dagger \Gamma_0 \Gamma^\mu \phi \lambda_k)_0 = \frac{1}{2} \sum_{a=1}^3 \bar{\psi}_a \Gamma_0 \Gamma^\mu \psi_a \lambda_k$$

corresponding to

$$\frac{1}{2} \sum_{a=1}^3 \bar{\Psi}_a \gamma_0 \gamma^\mu \Psi_a \lambda_k = \frac{1}{2} \text{tr} \bar{\Psi} \gamma_0 \gamma^\mu \Psi \lambda_k.$$

The Lagrangian  $L_0$  can be defined by

$$L_0 = (\phi^\dagger \Gamma_0 (e^5 \Gamma^\mu \partial_\mu - m) \phi)_0$$

in agreement with

$$L_0 = \text{tr}(\bar{\Psi} (i \gamma^\mu \partial_\mu - m) \Psi).$$

**PROOF.**

$$\phi^\dagger \Gamma_0 (e_5 \Gamma^\mu \partial_\mu - m) \phi = (e_1 e_0, e_2 e_0, e_3 e_0) \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{\psi}_3 \end{pmatrix} \Gamma_0 (e_5 \Gamma^\mu \partial_\mu - m) (\psi_1, \psi_2, \psi_3) \begin{pmatrix} e_1 e_0 \\ e_2 e_0 \\ e_3 e_0 \end{pmatrix}.$$

Again it is immediate that

$$(\phi^\dagger \Gamma_0 (e^5 \Gamma^\mu \partial_\mu - m) \phi)_0 = \sum_{a=1}^3 \bar{\psi}_a \Gamma_0 (e_5 \Gamma^\mu \partial_\mu - m) \psi_a$$

corresponding to

$$\sum_{a=1}^3 \Psi_a^\dagger \gamma_0 (i \gamma^\mu \partial_\mu - m) \Psi_a = \text{tr}(\bar{\Psi} (i \gamma^\mu \partial_\mu - m) \Psi). \quad \square$$

$L_1$  can be derived from  $L_0$  by replacing  $\partial_\mu$  by  $D_\mu$  in the usual way. For  $L_F$  one obtains the expression

$$L_F = -\frac{1}{2} \text{tr}(F_{\mu\nu} F^{\mu\nu}) = -\frac{1}{4} F_{\mu\nu}^k F_k^{\mu\nu}$$

with

$$F_{\mu\nu} = -\frac{1}{2} e_5 F_{\mu\nu}^k \lambda_k.$$

Observing that

$$\hat{\phi} = \phi U = \psi U \begin{bmatrix} e_1 e_0 \\ e_2 e_0 \\ e_3 e_0 \end{bmatrix} = \hat{\psi} \begin{bmatrix} e_1 e_0 \\ e_2 e_0 \\ e_3 e_0 \end{bmatrix}$$

it is a trivial task to prove the gauge invariances:

1.  $\hat{D}_\mu \hat{\phi} = (D_\mu \phi) U$
2.  $\hat{F}_{\mu\nu} = U^{-1} F_{\mu\nu} U$
3.  $\hat{L} = L.$

#### FINAL REMARKS

##### 1. *Electro-weak forces*

In this contribution we did not deal with the Glashow-Weinberg-Salam model of electro-weak forces and associated symmetry group  $SU(2) \times U(1)$  because all the necessary material can be found by combining Section 3 and [7].

##### 2. *Gravitational forces*

If one takes gravitational forces into account, Dirac's equation D1 transforms into the so-called Dirac-Weyl equation as treated by G.G.A. Bäuerle in these proceedings [9]. I hope, translation of this formalism into STA-language can be given in the near future.

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# Quark confinement in a model with induced metric

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## 1 INTRODUCTION

One of the striking features of meson spectroscopy is the close resemblance between the ‘charmonium’ and ‘beautonium’ spectra. With some imagination also a ‘strangonium’ spectrum can be discerned. Not only the ratios between the mass-level differences seem to be nearly the same for all ‘quarkonia’ spectra, but also their absolute values (see fig. 1). If mesons are considered as bound states of quarks and antiquarks one can consider ‘strangonium’ as a bound state of the strange quark  $s$  and its antiquark  $\bar{s}$ , ‘charmonium’ as a bound state of the charmed quark  $c$  and its antiquark  $\bar{c}$ , and ‘beautonium’ as a bound state of the beauty quark  $b$  and its  $\bar{b}$ . A direct comparison can be made with ‘positronium’, a bound state of an electron  $e^-$  and its antiparticle  $e^+$ , or ‘muonium’, a bound state of a  $\mu^-$  and a  $\mu^+$ . A comparison of the latter two ‘atoms’ shows that the discrete spectra are proportional to the mass, but otherwise equivalent. In the same way as this scaling behaviour can be understood in terms of properties of the Coulomb potential can one try to understand the scaling behaviour of the quarkonia spectra in terms of properties of the potential which binds the quarks together.

Let us assume that the spectra of the different quarkonium atoms can be obtained by solving a Schrödinger equation (which is a good approximation as long as the level differences are small compared to the rest masses themselves). Let

$$\left[ -\frac{1}{2m} \frac{d^2}{dr^2} + \frac{L^2}{2mr^2} + V(r,m) \right] \psi = E\psi, \quad (1)$$

where  $m$  is the reduced mass of the quarks  $q$  and  $\bar{q}$  bound within the

quarkonium system under consideration,  $E$  is the total energy,  $V(r, m)$  the potential in terms of  $m$  and the distance  $r$  between the quarks and  $\vec{L}$  the angular momentum (we take Planck's reduced constant  $\hbar$  and the light velocity  $c$  both equal to 1). For simplicity the spin of the quarks is not taken into account.

The question is which equation  $V(r, m)$  must satisfy in order for the spectra to have the desired scaling property. For that purpose we introduce the parameter  $\rho$ :

$$\rho = \sqrt{mr}. \quad (2)$$

This transforms eq. (1) into

$$\left[ -\frac{1}{2} \frac{d^2}{d\rho^2} + \frac{L^2}{2\rho^2} + \tilde{V}(\rho, m) \right] \psi = E\psi. \quad (3)$$

The condition that the spectra be independent of  $m$  except for an additional constant now becomes

$$\tilde{V}(\rho, m) = \tilde{V}_1(\rho) + \tilde{V}_2(m). \quad (4)$$

Next we find out how  $V(r, m)$  itself must depend on  $m$ . If the interactions between the quarks are **flavor-independent**, then  $\frac{\partial V}{\partial r}$  is independent of  $m$  (the forces do not depend on the quark species) and we have

$$V(r, m) = V_1(r) + V_2(m), \quad (5)$$

We can solve the equations (2), (4) and (5) and obtain

$$V(r, m) = \gamma \ln r + k(m), \quad (6)$$

where  $\gamma > 0$  is the condition for confinement. This potential is known as the potential of Quigg and Rosner [1] (see fig. 2 for the spectra). A variant is the potential of A. Martin [2]:

$$V = \gamma r^{0.1} + k(m), \quad (7)$$

which is nearly logarithmic.

Not only the congruence of the spectra is explained by such a potential, also the details of the spectra are quite reasonable. The only disadvantage is that it is impossible to find a fundamental reason why the potential should be logarithmic. Since the discovery of beautionium (also called bottomonium) in 1977 no connection with fundamental field theory has ever been found. Many authors therefore claim that the resemblance is just an accident.

Maybe a radical rethinking of what flavor independence means might establish a link between the quarkonia spectra and fundamental field theory. A very strong candidate for such a theory is Quantum Chromo Dynamics (QCD) in which quarks are endowed with 'color'-degrees of freedom and interact with a non-abelian gauge field, called the 'gluon'-field. Forces due to gluon exchange between quarks are independent of the quark mass and are therefore flavor-independent, at least in first instance, since renormalization causes the

coupling constant to become a ‘running’ coupling constant which is flavor-dependent. The latter is a small effect. Nevertheless, let us assume that the forces which bind the quarks are not **independent** of the quark masses, but **proportional** to the quark masses. Then, instead of eq. (5) we have

$$V(r, m) = mV_1(r) + V_2(m). \quad (8)$$

If we now solve eqs (2), (4) and (8), we find:

$$\left. \begin{array}{l} \tilde{V}(\rho, m) = C\rho^2 + \tilde{V}_2(m) \\ \text{and} \\ V(r, m) = mCr^2 + \tilde{V}_2(m) \end{array} \right\} \quad (9)$$

With  $C = \frac{1}{2}\omega^2$  we find

$$V(r, m) = \frac{1}{2}m\omega^2 r^2 + k(m), \quad (10)$$

which is the isotropic harmonic oscillator potential with universal frequency  $\omega$ . Fig. 3 shows the spectra.

Although the harmonic oscillator potential makes a much better chance to follow from fundamental field theory than the logarithmic potential, one has to explain why the forces are proportional to the quark masses and why the physical spectrum deviates from the ‘bare’ harmonic oscillator spectrum.

Note that when forces are proportional to the mass, accelerations are mass independent. Flavor independence of forces is replaced by flavor independence of accelerations. To reconcile this with the fact that gluon exchange gives rise to flavor independent forces, one must separate the forces due to perturbative QCD from those of nonperturbative QCD. Not much is known of the latter. In particular, it is not known how nonperturbative QCD can cause permanent confinement of quarks. Flavor independence of accelerations reminds one of gravitational forces. Also here the accelerations are more important than forces. Free particles follow timelike geodesics in a curved space-time and these geodesics are mass independent. This does not mean that masses are unimportant, because masses are sources of gravitation. Thus, although the sun and the planets all follow timelike geodesic trajectories in a curved space, the sun dominates the solar system because of its mass. We shall therefore try to give a geometric description of ‘quasi-free’ quark motion which only temporarily subdues the importance of quark masses. The model presented here is manifestly classical. A consistent formulation in terms of a Lagrangian field theory can be given and opens the door to quantization. A real scalar dynamical field  $\phi$  is put to work for ‘inducing’ an alternative metric field  $g_{\mu\nu}$ , called a quasi or induced metric.

## 2 THE HARMONIC OSCILLATOR AND THE ANTI-DE SITTER SPACE

Suppose that the harmonic oscillator spectrum be exact and that wave packets carry out exact harmonic oscillations with frequency  $\omega$ . The higher the energy, the larger the amplitude. let  $u$  be the displacement from the equilibrium position. Then

$$\dot{u}_{\max} = \omega u_{\max}. \quad (11)$$

The largest velocity is  $\omega$  times the largest displacement, but since the velocity can never surpass light velocity we have that  $u_{\max}$  can never surpass  $\frac{1}{\omega}$ . All oscillations take place within a sphere of radius  $R = \frac{1}{\omega}$ . An idea of the magnitude of  $R$  can be obtained by determining the average level spacing  $\Delta E$  of all quarkonia states. Then  $R \approx \frac{c\hbar}{\Delta E} \approx 1 \text{ fm} = 10^{-15} \text{ m}$ . The sphere of radius  $R$  can be visualized as a rigid 'bag', within which the quarks are confined. The heavier the quarks, the more they tend to concentrate in the center of the bag. Nevertheless, all mesons have the same size in our model. One could also think of the bag as wrapping a gluon cloud which must be nearly identical for all mesons.

Let us define the following quasi-metric tensor field  $g_{\mu\nu}$  inside a sphere of radius  $R$  [3]. (The true metric tensor field is still  $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ .)

$$\begin{aligned} g_{kl} &= -\frac{R^2}{R^2 - r^2} \left[ \delta^{kl} + \frac{x^k x^l}{R^2 - r^2} \right], \quad (k, l = 1, 2, 3), \\ g_{k0} &= g_{0k} = 0, \\ g_{00} &= \frac{R^2}{R^2 - r^2} \end{aligned} \quad (12)$$

where  $r^2 = \sum_k x^k x^k$ ,  $x^k$  being the position of a point inside the sphere. Fig. 4 gives an artists view of this metric by showing the quasi-unit spheres. Note that they become singular at the surface of the bag. Also, in the center the quasi metric tensor is equal to the metric tensor.

The quasi-geodesic equation is, with  $x^0$  being the time coordinate,

$$\frac{d^2 x^\mu}{d\bar{\tau}^2} + \bar{\Gamma}_{\nu\rho}^\mu \frac{dx^\nu}{d\bar{\tau}} \frac{dx^\rho}{d\bar{\tau}} = 0, \quad (13)$$

where

$$\bar{\Gamma}_{\nu\rho}^\mu = \frac{1}{2} g^{\mu\lambda} (\partial_\rho g_{\lambda\nu} + \partial_\nu g_{\lambda\rho} - \partial_\lambda g_{\nu\rho}) \quad (14)$$

is the quasi affine connection and  $\bar{\tau}$  the quasi proper time. Rewriting eq. (13) one finds, with  $x^0 = t$ :

$$\frac{d^2 x^k}{dt^2} = -\frac{1}{R^2} x^k, \quad (15)$$

Harmonic oscillator ( $\omega=1$ )	Lie algebra ( $\omega=1$ ) of $SO(1,2)^\dagger$
$[a, H] = a$ $[a^\dagger, H] = -a^\dagger$ $[a, a^\dagger] = 1$	$[a, H] = a$ $[a^\dagger, H] = -a^\dagger$ $[a, a^\dagger] = 1$
One vacuum state $ 0\rangle$ with $a 0\rangle=0$ and $\langle 0 0\rangle=1$	Two 'vacuum' states $ 0\rangle$ and $ 0'\rangle$ with 1. $a 0\rangle=0$ and $\langle 0 0\rangle=1$ $H(a^\dagger)^n 0\rangle = (E_0+n)(a^\dagger)^n 0\rangle$ $\langle 0 a^n(a^\dagger)^n 0\rangle > 0$ , provided $E_0 > 0$
$H(a^\dagger)^n 0\rangle = (E_0+n)(a^\dagger)^n 0\rangle$ $\langle 0 a^n(a^\dagger)^n 0\rangle > 0$	2. $a^\dagger 0'\rangle=0$ and $\langle 0' 0'\rangle=1$ $Ha^n 0'\rangle = -(E_0+n)a^n 0'\rangle$ $\langle 0' (a^\dagger)^n a^n 0'\rangle > 0$ , provided $E_0 > 0$ .

which is the isotropic harmonic oscillator equation with frequency  $\omega = \frac{1}{R}$ . Thus, all geodesics (quasi-timelike, -lightlike, or -spacelike) are at least parts of harmonic oscillations around one and the same space point with universal frequency  $\omega$ , but those oscillations which are confined to the interior of the sphere are quasi-timelike and can represent particle motion.

The metric (12) has a very high degree of symmetry. It is invariant under hyperbolic rotations of the group  $O(3,2)$  or rather the universal covering of it. It is called the (universal covering of the) anti-de Sitter group and the metric space is the anti-de Sitter space. If space reflections and time reversal are omitted one deals with the restricted group  $SO(3,2)^\dagger$ . The subgroup  $SO(3) \otimes SO(2)$  is important because  $SO(3)$  represents the spherical symmetry and  $SO(2)$  the invariance under time translations of the metric.

In order to see that this has something to do with harmonic oscillations we compare the spectrum generating algebra of the one dimensional harmonic oscillator with the Lie algebra of the group  $O(1,2)$  (see cadre).

In fig. 5 a scetch in made of the harmonic oscillator and  $O(1,2)$  spectrum. Note that the  $O(1,2)$  spectrum displays a kind of 'Dirac sea' of negative energy states, which is characteristic of a relativistic model. By increasing  $E_0$  one approaches more and more the nonrelativistic harmonic oscillator. This can be seen by making the following substitution

$$\begin{aligned}
H &= E_0 + \bar{H} \\
a &= \sqrt{E_0} \bar{a} \\
a^\dagger &= \sqrt{E_0} \bar{a}^\dagger
\end{aligned} \tag{16}$$

and by subsequently ignoring  $\bar{H}$  in the expression for  $[\bar{a}, \bar{a}^\dagger]$ . Then the Lie algebra transforms into the spectrum generating algebra of the harmonic oscillator.

There exists also a geometric way to show how a maximally symmetric space produces a quasi-metric within a sphere. Again we pass to the one-dimensional harmonic oscillator and consider a line element of length  $2R$  instead of a sphere. For this we consider fig. 6. In this figure the maximally symmetric space is represented by a hyperboloid. Geodesics are obtained by intersecting

this by a plane through the center. Quasi-timelike geodesics correspond to circles or ellipses. The latter are closed. In order to open them up the hyperboloid is replaced by a 'scroll' representing the universal covering space. Each point on the hyperboloid represents points on an infinite number of Riemann sheets.

In order to represent quasi-free particle motion on the 'scroll' by particle motion in our space we roll up our flat space time (2 dimensional in this case) into a cylinder and pass this through the neck of the hyperboloid. Next every point on the scroll is projected sheet by sheet on the cylinder by central projection. The plane which cuts out the ellipse on the hyperboloid now cuts out an ellipse on the cylinder. This in turn transforms into a harmonic oscillation when the cylinder is unrolled again. Since the projection lines cannot have angles smaller than  $45^\circ$  with the 1-axis, the ellipses on the cylinder are confined to a strip of width  $2R$  if  $R$  is the radius of the cylinder.

### 3 A 'SOFT-BAG' MODEL FOR CONFINEMENT

So far the bag has been considered as a rigid object containing a quasi metric field which confines quarks. In reality the bag should react to the presence of the quarks. In what follows a scetch will be given of how this could be achieved [4].

Suppose in our Minkowski space-time a real scalar field  $\phi$  is present. We can see this as a solution of an Euler-Lagrange equation following from the Lagrangian

$$\mathcal{L}_{(\phi)} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi). \quad (17)$$

A massive field with  $\phi^4$  interaction would have

$$V(\phi) = \frac{1}{2} \mu^2 \phi^2 + \frac{\lambda}{4!} \phi^4, \quad (18)$$

but much more complicated forms can be found, even singular ones. We could use  $\phi$  to construct a quasi metric, for example,

$$g_{\mu\nu} = \eta_{\mu\nu} + \alpha \partial_\mu \phi \partial_\nu \phi. \quad (19)$$

This  $g_{\mu\nu}$  must satisfy a number of conditions. It must have the same signature as  $\eta_{\mu\nu}$  and particles moving with quasi light velocity may never move faster than real light. This means that there should be a quasi light cone which always should be inside the real light cone (see fig. 7). Suppose  $p^\mu$  is tangent to the particle trajectory on the quasi light cone. Then

$$g_{\mu\nu} p^\mu p^\nu = 0. \quad (20)$$

Form eq. (19) it then follows that

$$\eta_{\mu\nu} p^\mu p^\nu + \alpha (p^\mu \partial_\mu \phi)^2 = 0. \quad (21)$$

Thus we see that  $\eta_{\mu\nu} p^\mu p^\nu \geq 0$  if  $\alpha \leq 0$ .

When  $\partial_\mu \phi = 0$  we find that  $g_{\mu\nu} = \eta_{\mu\nu}$  and there is no difference between the

Minkowski metric and the quasi metric. When  $\partial_\mu\phi$  is small, the  $g_{\mu\nu}$  and  $\eta_{\mu\nu}$  do not differ so much, but when  $\partial_\mu\phi$  is so large that

$$1 + \alpha\partial_\mu\phi\partial^\mu\phi = 0, \quad (22)$$

then the quasi light cone collapses into a line. If  $\partial_\mu\phi$  becomes even larger, then the signature changes such that the light cone disappears. Somehow the system should prevent  $\partial_\mu\phi$  from passing this point. There are more requirements which the metric must satisfy. If a collapse of the light cone takes place, it should be such that quasi free particles are prevented from passing that point. The ‘bag’, which is the set of space-time points in which collapse takes place, should act as an effective barrier for such particles (see fig. 8).

Now suppose that there exists another real scalar field  $\zeta$  with the following Lagrangian

$$\mathcal{L}_{(\zeta)} = \frac{1}{2} \sqrt{-g} g^{\rho\sigma} \partial_\rho\zeta\partial_\sigma\zeta - \frac{1}{2} \sqrt{-g} \mu^2 \zeta^2, \quad (23)$$

with  $g = \det g_{\mu\nu}$  and  $g^{\rho\sigma}$  the inverse of  $g_{\rho\sigma}$ . Then since the Euler-Lagrange equation for  $\zeta$  is linear in  $\zeta$ , so that the superposition principle holds, we have to do with a free field in some sense. We shall call it quasi free and indeed, if a wave packet is constructed it will follow more or less a quasi timelike geodesic trajectory with respect to the metric  $g_{\mu\nu}$ . Such a wave packet cannot pass the wall of the bag and is therefore confined.

How can we construct a Poincaré-invariant model? Write

$$\mathcal{L} = \mathcal{L}_{(\phi)} + \mathcal{L}_{(\zeta)}. \quad (24)$$

If this is supplemented with the form of  $V(\phi)$  and the form of  $g_{\mu\nu}$  as a function of  $\eta_{\mu\nu}$  and  $\partial_\rho\phi$ , then we have in fact a model in which  $\mathcal{L}$  is a function of  $\eta_{\mu\nu}$  (trivial),  $\phi$ ,  $\partial_\mu\phi$ ,  $\zeta$ , and  $\partial_\mu\zeta$  and these produce the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\rho \frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}, \quad (25)$$

and

$$\frac{\partial\mathcal{L}}{\partial\zeta} = \partial_\rho \frac{\partial\mathcal{L}}{\partial(\partial_\rho\zeta)}, \quad (26)$$

Because  $\mathcal{L}_{(\zeta)}$  contains  $\partial_\rho\phi$ , eq. (25) differs from the equation

$$\frac{\partial\mathcal{L}_{(\phi)}}{\partial\phi} = \partial_\rho \frac{\partial\mathcal{L}_{(\phi)}}{\partial(\partial_\rho\phi)},$$

which is not valid anymore. The latter would lead to a rigid bag, impervious to the motion of its contents, while eq. (25) reacts to the presence of the field  $\zeta$ , which means that we deal with a soft bag. Since  $\mathcal{L}_{(\phi)}$  does not contain  $\zeta$  or  $\partial_\rho\zeta$ , the field  $\zeta$  listens only to the metric  $g_{\mu\nu}$ . Thus  $\zeta$  helps to shape  $g_{\mu\nu}$ , but once  $g_{\mu\nu}$  is defined, the dynamics of  $\zeta$  are completely subjected to  $g_{\mu\nu}$ . Note also that  $\mathcal{L}$  never contains second or higher order derivatives of the fields  $\phi$  and  $\zeta$  and that is how it should be.

Note that except for  $\eta_{\mu\nu}$  all fields occurring in  $\mathcal{L}$  are dynamical fields, *e.g.* no a priori defined nontrivial functions of  $x^\mu$  are present in the Lagrangian. Stated differently, the  $\mathcal{L}$  is a completely Poincaré invariant expression in terms of the dynamical fields. This leads to conservation of total energy, momentum and angular momentum. A stress-energy-momentum tensor density can be defined as follows. First rewrite  $\mathcal{L}_{(\phi)}$  and  $\mathcal{L}_{(\zeta)}$ ;

$$\mathcal{L}_{(\phi)} = \frac{1}{2} \sqrt{-\eta} \eta^{(\mu\nu)} \partial_\mu \phi \partial_\nu \phi - \sqrt{-\eta} V(\phi), \quad (\eta = \det \eta_{\mu\nu}), \quad (27)$$

with indices between brackets meaning symmetrization (in this case only a formal operation), and

$$\mathcal{L}_{(\zeta)} = \frac{1}{2} \sqrt{-g} g^{(\rho\sigma)} \partial_\rho \zeta \partial_\sigma \zeta - \frac{1}{2} \sqrt{-g} \mu^2 \zeta^2, \quad (28)$$

and substitute this in the expression for  $\mathcal{L}$ . Then determine

$$T^{\mu\nu} = T^{\nu\mu} = -2 \frac{\partial \mathcal{L}}{\partial \eta_{\mu\nu}}. \quad (29)$$

The tensor  $T^{\mu\nu}$  then satisfies the continuity equation

$$\partial_\nu T^{\mu\nu} = 0. \quad (30)$$

The factor -2 is chosen to make  $T^{\mu\nu}$  equal to the stress-energy momentum tensor (density). The symmetry of  $T^{\mu\nu}$  and the validity of the continuity-equation guarantee conservation of energy, momentum and angular momentum. We can now write

$$T^{\mu\nu} = -2 \left[ \frac{\partial \mathcal{L}}{\partial \eta_{\mu\nu}} \Big|_{g^{\rho\sigma}} + \frac{\partial g_{\rho\sigma}}{\partial \eta_{\mu\nu}} \frac{\partial \mathcal{L}}{\partial g_{\rho\sigma}} \Big|_{\eta_{\mu\nu}} \right] = T_{(\phi)}^{\mu\nu} + T_{(\zeta)}^{\mu\nu}, \quad (31)$$

where

$$T_{(\phi)}^{\mu\nu} = -2 \frac{\partial \mathcal{L}}{\partial \eta_{\mu\nu}} \Big|_{g^{\rho\sigma}} \quad (32)$$

is the stress-energy-momentum tensor of the  $\phi$ -sector (the 'bag') only, and

$$T_{(\zeta)}^{\mu\nu} = -2 \frac{\partial g_{\rho\sigma}}{\partial \eta_{\mu\nu}} \frac{\partial \mathcal{L}}{\partial g_{\rho\sigma}} \Big|_{\eta_{\mu\nu}} = \frac{\partial g_{\rho\sigma}}{\partial \eta_{\mu\nu}} \Gamma^{\rho\sigma} \quad (33)$$

is the stress-energy-momentum tensor of the  $\zeta$ -sector (the confined field). The force balance can be written in form

$$\partial_\nu T_{(\zeta)}^{\mu\nu} = -\partial_\nu T_{(\phi)}^{\mu\nu}, \quad (34)$$

and keeps track of the flow of energy, momentum and angular momentum from the confined field towards the bag and vice versa.

The symmetric tensor density  $\Gamma^{\rho\sigma}$  is the quasi stress-energy-momentum tensor density of the confined field. It leads to the notion of quasi energy and



quasi momentum, which according to eq. (33) may differ substantially from the real energy and momentum of the  $\zeta$  sector. The quasi energy and quasi momentum are not conserved but locally conserved in the sense that

$$\bar{D}_\nu \Gamma^{\mu\nu} = 0, \quad (35)$$

where  $\bar{D}_\nu$  denotes the quasi covariant derivative, *i.e.*

$$\bar{D}_\nu \Gamma^{\mu\nu} = \partial_\nu \Gamma^{\mu\nu} + \bar{\Gamma}^\mu_{\rho\nu} \Gamma^{\rho\nu} + \bar{\Gamma}^\nu_{\rho\nu} \Gamma^{\mu\rho}. \quad (36)$$

From eqs (17), (32), (33) and (35) it follows that, with  $\square = \partial_\mu \partial^\mu$ .

$$\partial^\mu \phi [\square \phi + \mathcal{V}(\phi)] = -\partial_\nu \left[ \frac{\partial g_{\rho\sigma}}{\partial \eta_{\mu\nu}} \Gamma^{\rho\sigma} \right]. \quad (37)$$

This will be valid if  $\Gamma^{\rho\sigma}$  satisfies eq. (35). Thus, the right hand side of eq. (37) must contain the factor  $\partial^\mu \phi$  which then can be divided out. The result is

$$\square \phi + \mathcal{V}(\phi) = \text{source term}. \quad (38)$$

The source term not only depends on the confined fields, but also on  $\phi$  itself. This scheme allows the contents of the bag to exert a pressure on the bag wall.

It is interesting to note that point particles can take the place of fields. Let

$$\bar{\Gamma}^{\mu\nu} = m \frac{u^\mu u^\nu}{u^0} \delta^{(3)}[\vec{x} - \vec{x}_0(t)], \quad (39)$$

where  $\vec{x} = \vec{x}_0(t)$  is a given particle world line,  $g_{\mu\nu} u^\mu u^\nu = 1$ ,  $u^0 > 0$ ,  $m > 0$  and  $u^\mu$  is tangent to the curve  $\vec{x} = \vec{x}_0(t)$ . Then if  $\vec{x}_0(t)$  is a quasi timelike geodesic trajectory we have

$$\bar{D}_\nu \bar{\Gamma}^{\mu\nu} = 0. \quad (40)$$

Thus there is quasi local conservation of energy and momentum. True energy and momentum are not conserved but exchanged with the bag proper. A point source appears in eq. (38).

Let us next elaborate on the shape of  $\mathcal{V}(\phi)$  in order to produce bags. Consider a point particle with equation of motion

$$M\ddot{x} + U(x) = 0, \quad (41)$$

where  $U(x)$  is some smooth potential with a spike at  $x=0$  of finite height  $\Delta U$  (see fig. 8). Suppose a particle is put on top of the spike. Although in unstable equilibrium, it may stay there forever. It may be there also for all times  $< t_0$  and then rolling off, either to the left or to the right. When it rolls off to the left it disappears forever. When it rolls off to the right it climbs until it reaches point  $P$ , then it runs back and comes to rest forever on the top of the spike (fig. 9). It may also periodically leave the top and come back (fig. 10).

As a special case we may consider a linear potential with a spike. Then the particle trajectory consists of a segment of a parabola in between straight lines stretching out to infinity, or an infinite number of identical segments of parabolas interconnected by straight lines. In all cases we have

$$\Delta U = \frac{1}{2} m \dot{x}^2 \Big|_{t \downarrow t_0} . \quad (42)$$

When  $U(x) = \rho x + \text{spike}$ , one finds that the total width  $\Delta T$  of each segment is equal to

$$\Delta T = \sqrt{\frac{8m\Delta U}{\rho^2}} . \quad (43)$$

Next consider the equation

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + V(\phi) = 0. \quad (44)$$

If we are interested in static solutions this reduces to

$$\frac{\partial^2 \phi}{\partial x^2} - V(\phi) = 0. \quad (45)$$

Thus if

$$V(\phi) = -\rho\phi + \text{dip of depth } \Delta V \quad (46)$$

(see fig. 11), and if  $\phi$  satisfies the boundary condition

$$|\phi| < c < \infty \quad \text{for } \phi \rightarrow \pm\infty, \quad (47)$$

then the solutions are the same as those of figs 9 and 10 with segments of parabolas, but now  $x$  replaced by  $\phi$  and  $t$  by  $x$ . By carefully combining two limiting processes (the static limit and the limit of an infinitely thin spike) one finds solutions consisting of irregularly spaced identical segments of parabolas. In the case of three space dimensions one similarly finds static spherically symmetric solutions of uniform size. These are the 'bags' we are looking for. Inside the bags the field  $\phi$  forms segments of paraboloids, while in the space between the bags the field  $\phi$  is zero. Note that at the bag surface we have, with  $\Delta V$  being the depth of the dip:

$$\partial_\rho \phi \partial^\rho \phi = -2\Delta V. \quad (48)$$

Of course, nonstatic solutions of nonspherical size also exist. They all are characterized by the existence of singular surfaces and regions where  $\phi$  vanishes. At the singular surface  $\partial_\rho \phi \partial^\rho \phi$  assumes a fixed value given by eq. (48).

Finally there is the question of the relation between  $g_{\mu\nu}$ ,  $\eta_{\mu\nu}$  and  $\phi$ . First of all, we would like  $g_{\mu\nu}$  to become singular on the bag wall. Thus it must become singular when eq. (48) is satisfied. Then it should have the right signature, it should not violate causality and it should indeed act as an effective barrier for quasi free particles or fields, massive or massless, when  $g_{\mu\nu}$  is singular. It may however give rise to a 'sticky' bag wall. Particles approaching the wall get glued to it as if they are falling towards a black hole. Also this must be prevented. Keeping in mind that second and higher order derivatives of  $\phi$  are not allowed to occur in  $g_{\mu\nu}$  one finds for the simplest functional form of the quasi metric tensor the inverse of

$$g^{\mu\nu} = (1 + \alpha \partial_\rho \phi \partial^\rho \phi) \cdot (\eta^{\mu\nu} + \alpha \partial^\mu \phi \partial^\nu \phi), \quad (49)$$

with

$$\alpha = \frac{1}{2\Delta V} > 0. \quad (50)$$

For static, spherically symmetric solutions this leads to the form (12) with

$$R = \frac{3\sqrt{2\Delta V}}{|\rho|}. \quad (51)$$

#### 4 THE QUESTION OF HIGHER DERIVATIVES

Can higher derivatives always be prevented? Suppose we had a vector field  $A_\mu$  whose Lagrangian in flat space-time were given by

$$\mathcal{L} = -\frac{1}{2}(\partial_\nu A_\mu)(\partial^\nu A^\mu) + \frac{1}{2}\mu^2 A_\mu A^\mu, \quad (52)$$

then in a curved space-time this should be written as

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}g^{\alpha\beta}g^{\gamma\delta}(\bar{D}_\gamma A_\alpha)(\bar{D}_\delta A_\beta) + \frac{1}{2}\sqrt{-g}\mu^2 g^{\alpha\beta}A_\alpha A_\beta. \quad (53)$$

The  $\bar{D}_\gamma$  introduces second derivatives of  $\phi$  and so this  $\mathcal{L}$  is not allowed. However, if we construct  $\mathcal{L}$  in the way it should be done:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mu^2 A_\mu A^\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (54)$$

then this gives for a curved space-time

$$\mathcal{L} = -\frac{1}{4}\sqrt{-g}g^{\alpha\beta}g^{\gamma\delta}F_{\alpha\gamma}F_{\beta\delta} + \frac{1}{2}\sqrt{-g}g^{\alpha\beta}A_\alpha A_\beta, \quad (55)$$

which does not contain second derivatives of  $\phi$  and which is therefore allowed. However, a quasi-covariant 'gauge fixing term' of the form

$$-\frac{1}{2}\lambda(g^{\rho\sigma}\bar{D}_\rho A_\sigma)^2 \quad (56)$$

is not allowed. This is not fatal, because there exist other gauge fixing possibilities.

Next, the quasi curvature tensor or its contracted forms are forbidden to enter. They clearly introduce higher derivatives of  $\phi$ .

Very special is the role of spinor fields. Here it seems as if the quasi affine connection is inevitable. We have in flat space-time

$$\mathcal{L}_{(\psi)} = \frac{i}{2}[\bar{\psi}\gamma^\mu\partial_\mu\psi - (\partial_\mu\bar{\psi})\gamma^\mu\psi] - m\bar{\psi}\psi. \quad (57)$$

Its form in curved space-time is rather complicated and requires the introduction of so-called tetrad or vierbein fields. Somehow  $\partial_\mu$  must be replaced by a  $\bar{D}_\mu$  which involves these auxiliary fields and the quasi affine connection. By a unique stroke of luck all second derivatives of  $\phi$  drop out, provided  $\phi$  is the

only field present in  $g_{\mu\nu}$  and provided the real curvature tensor does not occur in  $g_{\mu\nu}$  in any generalization of  $\eta_{\mu\nu}$  to real curved space if one likes to do so. Under all circumstances minimal substitution of gauge fields, abelian or non-abelian, is allowed.

All taken together we see that most field Lagrangians fail to satisfy the criteria. Also it seems that more than one field  $\phi$  is not allowed for constructing bags. However, scalar fields can be confined. More important, abelian or non-abelian gauge fields minimally coupled to spinor fields, are all allowed. Thus, the QCD-Lagrangian can be written in  $g_{\mu\nu}$  language without causing trouble. Apparently, quarks and gluons can be confined within bags, and quarks can be given an electric charge. Moreover, quarks with large quasi rest masses dominate the scene.

#### 5 THE QUESTION OF STABILITY

According to Derrick's Theorem [5], solutions of the kind we have constructed are solutions which cannot be stable. This is reflected in the fact that  $V(0)$  is not the lowest value of  $V(\phi)$ . It might mean that unfilled bags cannot exist, but the question goes deeper. Consider eq. (38). In order to achieve stability one might introduce sources and simultaneously modify  $V(\phi)$  such that, at least for static spherically symmetric solutions,  $\phi$  remains unchanged. If  $V(\phi)$  can be modified such that  $V(0)$  is always smaller than  $V(\phi)$  for  $\phi \neq 0$ , then stability is obtained. So far, no sources have been found which can achieve this. What can happen though is that the system becomes too stable and returns to the hard bag model. Thus stability is still a problem. We are however in good company: also classical electrodynamics is unstable unless point charges are given an infinite mass. There the problems are reduced to manageable proportions by quantization, regularization and renormalization. So it must be hoped that quantization helps to stabilize bags.

Quantization might also throw light on the meaning of the field  $\phi$ . In this model it is treated as a fundamental field, like a Higgs field, but it may well be a condensate of already existing fields. As such it is then a pure quantum phenomenon.

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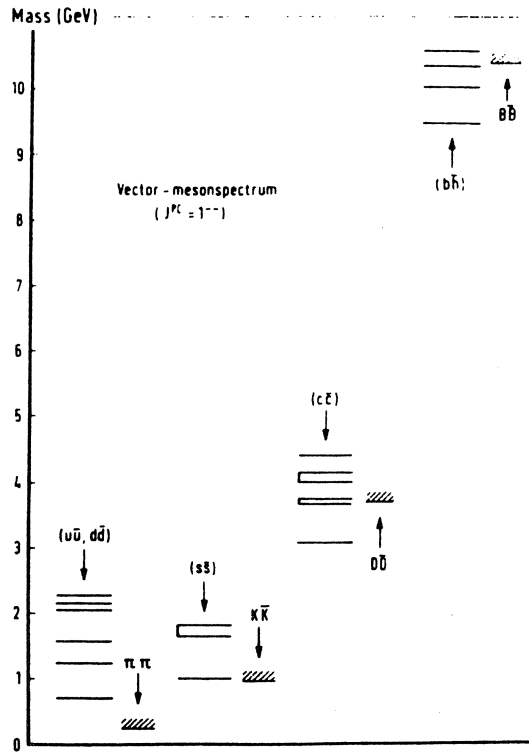


Figure 1: Mass spectrum of neutral vector mesons

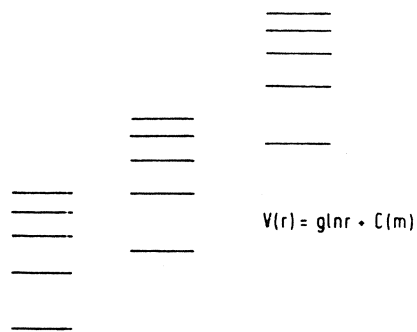


Figure 2: Mass spectrum of mesons in the model of Quigg and Rosner

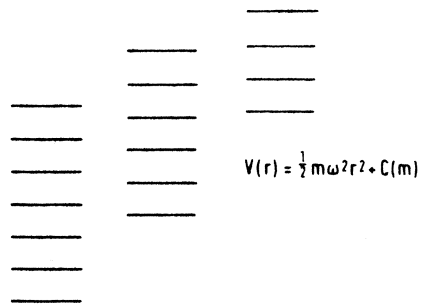


Figure 3: 'Bare' mass spectrum in harmonic oscillator model

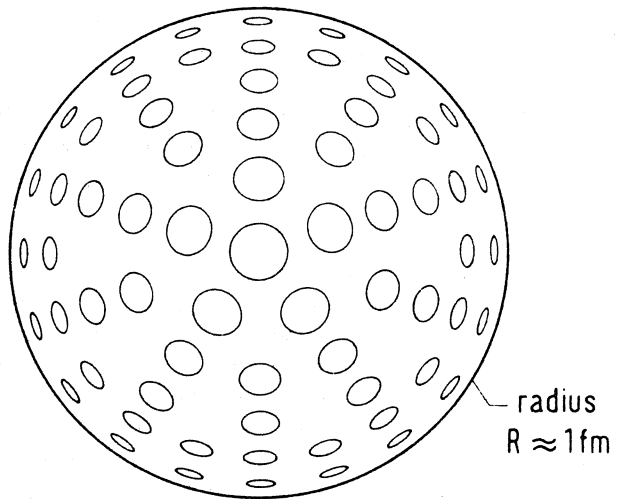


Figure 4: Static, spherically symmetric bag model with quasi unit spheres

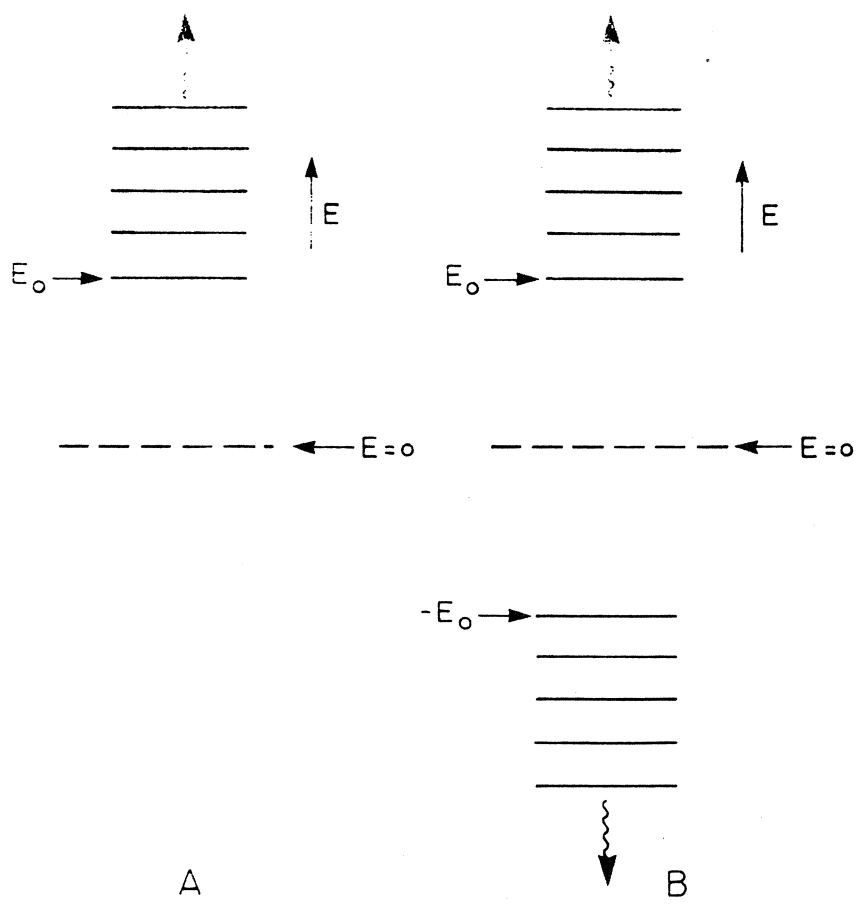
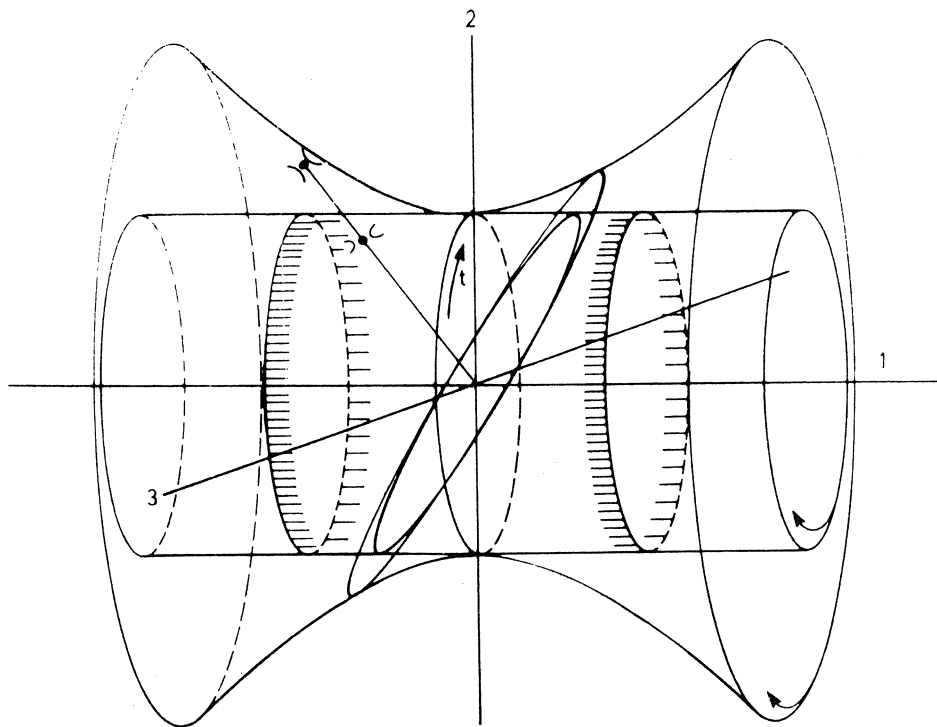


Figure 5: A. One dimensional harmonic oscillator spectrum  
 B. SO(1,2) spectrum



SO(1,2) bag  
 $g_{\mu\nu} = \text{diag}(-1, +1, +1)$

Figure 6: Geometric construction of quasi-free quark motion in a 2-dimensional space-time



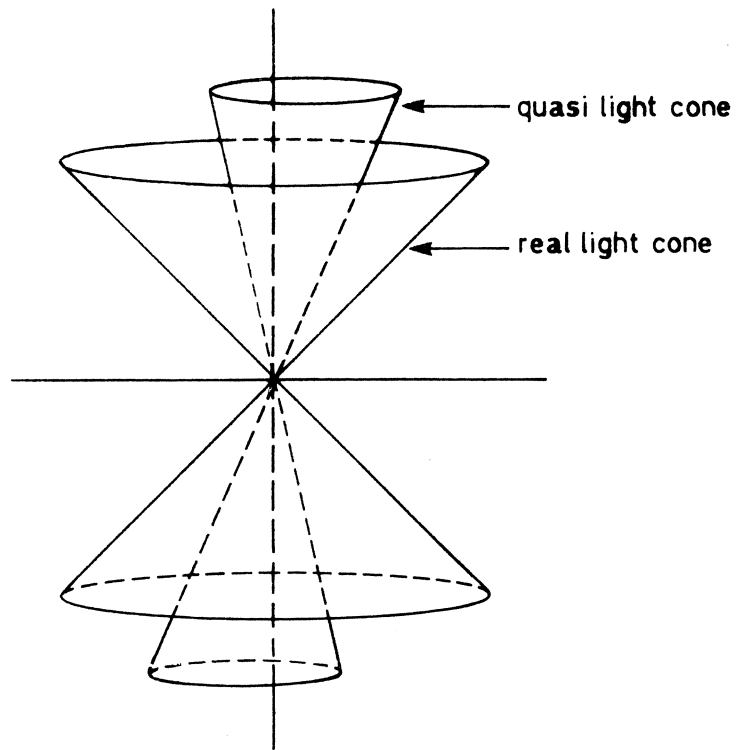


Figure 7: Relative positions of real and quasi lightcones

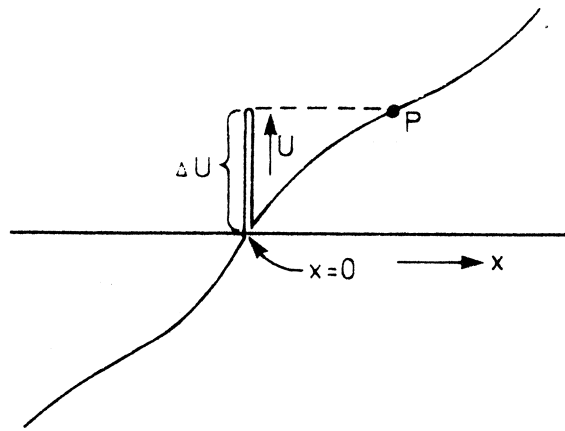


Figure 8: Particle potential with 'spike'

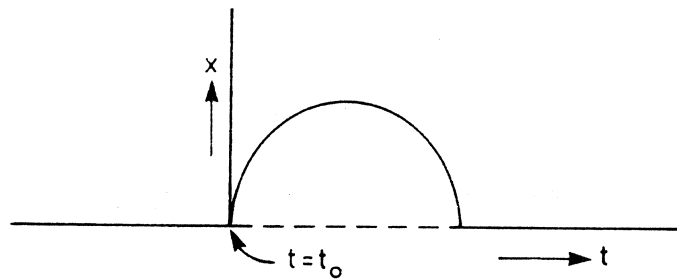


Figure 9: Particle motion for potential of fig. 8: one soliton

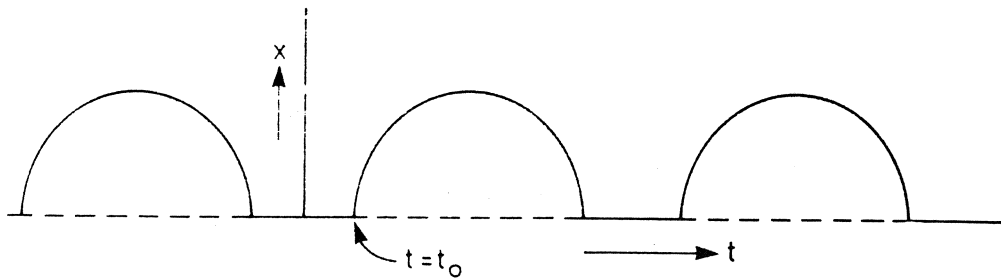


Figure 10: particle motion for potential of fig. 8: infinitely many solitons

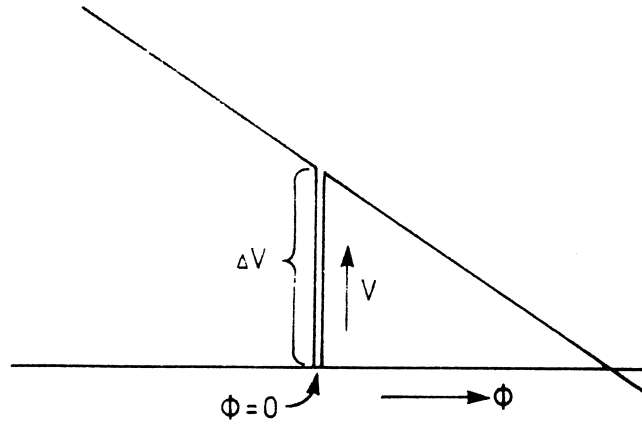


Figure 11: Linear potential with dip



# Strings and the Frenkel–Kac–Segal mechanism

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## 1. INTRODUCTION

The Frenkel–Kac–Segal mechanism is interesting both from a mathematical and a physical point of view. Its mathematical investigation was initiated by Frenkel and Kac [2] and Segal [11]. It gives there a so-called untwisted vertex operator realization of the basic representations of the simply-laced affine Kac–Moody algebras (see e.g. [3]). In physics it made its appearance through Halpern [8] (see also [6]). Here we will discuss, following [7], how it gives rise to gauge fields in a bosonic string theory. This is called the Frenkel–Kac–Segal mechanism. It provides a way to introduce the fundamental interactions of nature as gauge interactions without assuming the existence of gauge fields from the outset. Moreover, there are only two possible choices for the gauge group:  $E_8 \times E_8$  or  $\text{Spin}(32)/\mathbb{Z}_2$ .

In section 2 the bosonic string in Minkowski spacetime is reviewed. It is treated similar to the description of a relativistic particle by means of Hamilton’s action principle. For the action of the classical bosonic string the Nambu–Goto action and the Polyakov action are discussed. For the Nambu–Goto action the bosonic string is quantized via the so-called old-covariant approach. Its state vector space is constructed similar to that of the harmonic oscillator via creation and annihilation operators. Both the cases of open and closed strings are considered.

Section 3 deals with the situations where symmetries in a classical theory are not preserved in the act of quantizing the theory. Such circumstances are dubbed anomalies. Here their appearance is treated in the setting of Feynman’s functional integral quantization following Fujikawa (see e.g. [4] and [5]). In order to retain conformal invariance in the quantized bosonic string theory one is lead to put the dimension of Minkowski spacetime equal to 26.

Since spacetime appears to be four-dimensional 22 of these dimensions have to be made unobservable. In section 4 this is effected by toroidal compactification. Its discussion is continued in section 5 where a special case of toroidal compactification is considered. This gives rise to two possible affine Kac–Moody algebras as spectrum generating algebras.

## 2. BOSONIC STRING

In this section we discuss the classical as well as the quantum theory of a bosonic string in the setting of the special theory of relativity. Spacetime of special relativity is a four-dimensional Minkowski space. In elementary texts this space is introduced in a simple (coordinate dependent) way, namely as  $\mathbf{R}^4$  with a scalar product  $v \cdot w := v^0 w^0 - v^1 w^1 - v^2 w^2 - v^3 w^3$  ( $v, w \in \mathbf{R}^4$ ).

As a stepping stone towards the classical bosonic string we give a discussion of a particle in Minkowski spacetime. String theory will ultimately include gravity, and thus spacetime will be curved. It is for the sake of simplicity that we start with Minkowski spacetime.

*Spacetime* The primitive concept of the theory of relativity is that of an event. This arises as follows. Every physical phenomena occurs somewhere in space and happens in a certain stretch of time. One can imagine this phenomena to be partitioned as a collection of happenings for which the extensions in space and durations in time are negligible for an accurate description of this phenomena. Such an ‘infinitesimal’ happening is called an event. Spacetime  $M$  is defined to be the set of all (possible) events. It is assumed that spacetime can be equipped with coordinate systems. More precisely, spacetime is assumed to be a differentiable manifold. In view of later applications the spacetime manifold is supposed to be  $n$ -dimensional, although the immediately observable world strongly suggests the value  $n = 4$ . The successive events in the history of a particle gives rise to a one-dimensional submanifold of spacetime, the worldline of the particle.

The spacetime manifold of the theory of special relativity has more structure, it is assumed to be an  $n$ -dimensional Minkowski space  $\mathbf{M}$ , i.e. an  $n$ -dimensional affine space with a flat metric  $\eta$  with signature  $(1, -1, \dots, -1)$ .

The fact that  $\mathbf{M}$  is an affine space means that  $\mathbf{M}$  is equipped with a family of coordinate systems  $\{(\kappa_\alpha, \mathbf{M}) \mid \alpha \in I\}$ , where the coordinate maps  $\kappa_\alpha$  are defined on all of  $\mathbf{M}$  ( $I$  is an index set) and  $\kappa_\alpha(\mathbf{M}) = \mathbf{R}^n$ . Moreover, all coordinate transformations  $\kappa_\beta \circ \kappa_\alpha^{-1}: \mathbf{R}^n \rightarrow \mathbf{R}^n$  are assumed to be (possibly inhomogeneous) linear transformations. Hence, denoting the coordinates of  $x \in \mathbf{M}$  with respect to the charts  $\kappa_\alpha$  and  $\kappa_\beta$  respectively by  $\kappa_\alpha(x) = (x^0, x^1, \dots, x^{n-1}) \equiv (x^\mu)$  and  $\kappa_\beta(x) = (x^{0'}, x^{1'}, \dots, x^{n-1}') \equiv (x^{\mu'})$  one has

$$x^{\mu'} = L^{\mu'}_{\nu} x^{\nu} + a^{\mu'} \quad (\det(L^{\mu'}_{\nu}) \neq 0) \quad (1)$$

The coordinate systems  $(\kappa_\alpha, \mathbf{M})$  are called rectilinear and the coordinate transformations (1) are called affine. In an affine space the concept of a straight line makes sense. Indeed, a straight line is by definition a curve  $\gamma: \lambda \in \mathbf{R} \mapsto \mathbf{M}$  with linear coordinate expressions  $\kappa_\alpha \circ \gamma: \lambda \mapsto (v^0 \lambda + c^0, v^1 \lambda + c^1, \dots, v^{n-1} \lambda + c^{n-1})$  with not all  $v^\mu$  equal to zero. The whole point of the assumption that spacetime  $\mathbf{M}$  of special relativity is an affine space is that it allows to represent the worldlines of free particles in this theory by straight lines.

Since the metric tensor field  $\eta$  is taken to be flat in special relativity, the components of  $\eta$  with respect to each rectilinear coordinate system form a constant symmetric  $n \times n$  matrix with signature  $(1, -1, \dots, -1)$ . By means of an affine coordinate transformation this matrix can be diagonalized, i.e. there exists a rectilinear coordinate system  $(\kappa, \mathbf{M})$  such that the components of the metric have the form

$$\eta_{\mu\nu} = \eta(\partial_\mu|_p, \partial_\nu|_p) = \text{diag}(1, -1, \dots, -1) \quad (2)$$

where the coordinates of a point  $x \in \mathbf{M}$  are denoted by  $\kappa(x) = (x^0, x^1, \dots, x^{n-1}) \equiv (x^\mu) \in \mathbf{R}^n$  and  $\{\partial_\mu|_p\}_{\mu=0}^{n-1}$  is the coordinate basis in the tangent space  $T_p(\mathbf{M})$  for the rectilinear chart  $(\kappa, \mathbf{M})$ . Such a coordinate system  $(\kappa, \mathbf{M})$  is called a Lorentz coordinate system or Lorentz chart. Coordinate transformations between Lorentz coordinate systems are inhomogeneous Lorentz transformations (Poincaré transformations). The only coordinate systems we will use here on Minkowski spacetime are Lorentz coordinate systems. Instead of the zeroth coordinate  $x^0$  we also write  $ct$  ( $c$  speed of light) and  $t$  is called time or time coordinate of the event  $x \in \mathbf{M}$  with respect the Lorentz chart  $(\kappa, \mathbf{M})$ .

The Minkowski scalar product of a pair of vectors  $v, w \in T_p(\mathbf{M})$  is denoted by  $v \cdot w \equiv \eta(v, w) = \eta_{\mu\nu} v^\mu w^\nu$  (summation convention) and, in particular,  $v^2 \equiv v \cdot v$ . We use the notational convention that Greek indices run through the values  $0, 1, \dots, n-1$  and that Roman indices run through the values  $1, \dots, n-1$ .

*Free massive particle* It is assumed that the successive events in the history of a point particle form a one-dimensional submanifold  $\ell \subset \mathbf{M}$  diffeomorphic to the real numbers  $\mathbf{R}$ . The whole history of this particle is described by this submanifold  $\ell$ , the worldline of the particle. Each diffeomorphism between  $\ell$  and  $\mathbf{R}$  parametrizes  $\ell$  and thus gives rise to a curve

$$\gamma: \lambda \in \mathbf{R} \mapsto \gamma(\lambda) \in \ell \subset \mathbf{M} \quad (3)$$

in spacetime. Reparametrization of the curve (3) by means of a diffeomorphism  $f: \lambda \in \mathbf{R} \mapsto \tilde{\lambda} = f(\lambda) \in \mathbf{R}$  gives a new curve  $\tilde{\gamma}$  defined by  $\tilde{\gamma}(\tilde{\lambda}) := \gamma(\lambda)$ , i.e.  $\tilde{\gamma} = \gamma \circ f^{-1}$ . However, this new curve describes the same history of the particle since it involves the same events. That is, the curves  $\gamma$  and  $\tilde{\gamma}$  give rise to the same worldline  $\ell$ . It is to be stressed that the (physical) history of a particle is independent of the choice of a parametrization of its worldline.

Let  $(\mathbf{M}, \kappa)$  be a Lorentz chart, then the coordinate expression of the curve  $\gamma$ , i.e.  $\kappa \circ \gamma: \lambda \in \mathbf{R} \mapsto \kappa(\gamma(\lambda)) \in \mathbf{R}^n$ , will be denoted by  $x^\mu = x^\mu(\lambda)$ . A particle is called massive if all the tangent vectors  $\dot{\gamma}$  to its worldline are timelike vectors, i.e.

$$\eta(\dot{\gamma}, \dot{\gamma}) = \eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} > 0 \quad (\lambda \in \mathbf{R}) \quad (4)$$

A particle is called massless if  $\eta(\dot{\gamma}, \dot{\gamma}) = 0$  for all  $\lambda \in \mathbf{R}$ . A very useful parameter along the worldline of a massive particle is obtained from the arclength. The arclength  $s_{pq}^{(\gamma)}$  along the curve  $\gamma$  between the points  $p = \gamma(\lambda_1)$  and  $q = \gamma(\lambda_2)$  on this curve is defined by

$$s_{pq}^{(\gamma)} := \int_{\lambda_1}^{\lambda_2} \sqrt{\eta(\dot{\gamma}, \dot{\gamma})} d\lambda = \int_{\lambda_1}^{\lambda_2} \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda \quad (5)$$

Observe that the arclength of a segment of a worldline does not change under a reparametrization of this worldline. Let  $p_0 = \gamma(\lambda_0)$  be a fixed point on  $\gamma$ , then the parameter  $s = s(\lambda)$  assigned to the point  $x = \gamma(\lambda)$  is by definition the arclength  $s(\lambda) = s_{p_0x}^{(\gamma)}$ . For a massive particle we will often use such a parameter  $s$  to parametrize its worldline, and its coordinate expression then reads  $x^\mu = x^\mu(s)$ .

We now want to obtain, in a plausible way, the action  $S$  of a free massive particle (for the part of its worldline  $\ell$  between the events  $p$  and  $q$  on  $\ell$ ). Let  $x^\mu = x^\mu(\lambda)$  be a coordinate expression of a curve representing  $\ell$  such that  $\kappa(p) = (x^\mu(\lambda_1))$  and  $\kappa(q) = (x^\mu(\lambda_2))$ . The action  $S$  is a functional of  $x^\mu = x^\mu(\lambda)$  and a function of the parameters  $\lambda_1$  and  $\lambda_2$  of the endpoints  $p$  and  $q$ :

$$S = S_{\lambda_1\lambda_2}[x^\mu] \quad (6)$$

The role of  $S$  in classical physics is that it gives, via Hamilton's action principle, the equations of motion of the particle. Hamilton's action principle asserts the actual trajectory  $x^\mu = x^\mu(\lambda)$  of the particle for given endpoints  $x^\mu = x^\mu(\lambda_1)$  and  $x^\mu = x^\mu(\lambda_2)$  stands out from the collection of all other imaginable trajectories with the same endpoints by the property that  $S$  is stationary at  $x^\mu = x^\mu(\lambda_1)$ , i.e.

$$\left. \frac{d}{d\epsilon} S[x^\mu + \epsilon y^\mu] \right|_{\epsilon=0} = 0 \quad (7)$$

for all  $y^\mu = y^\mu(\lambda)$  with  $y^\mu(\lambda_1) = y^\mu(\lambda_2) = 0$  [all competing trajectories have the same endpoints!]. Let  $r$  be an event on  $\ell$  between  $p$  and  $q$  with  $\kappa(r) = (x^\mu(\lambda_3))$  ( $\lambda_1 < \lambda_3 < \lambda_2$ ), then the actions  $S_{\lambda_1\lambda_2}[x^\mu]$  and  $S_{\lambda_1\lambda_3}[x^\mu]$  have to give the same equations of motion for  $\lambda \in [\lambda_1, \lambda_3]$ . This can be implemented by requiring the action to be additive:

$$S_{\lambda_1\lambda_2}[x^\mu] = S_{\lambda_1\lambda_3}[x^\mu] + S_{\lambda_3\lambda_2}[x^\mu] \quad (8)$$

Taking  $n$  intermediate values  $\lambda_3 < \lambda_4 < \dots < \lambda_{n+2}$  between  $\lambda_1$  and  $\lambda_2$  instead of only  $\lambda_3$  this leads in the limit  $n \rightarrow \infty$  to

$$S_{\lambda_1\lambda_2}[x^\mu] = \int_{\lambda_1}^{\lambda_2} \mathcal{L} d\lambda \quad (9)$$

where

$$\mathcal{L} = \mathcal{L}(x^\mu, \frac{dx^\mu}{d\lambda}, \frac{d^2x^\mu}{d\lambda^2}, \dots) \quad (10)$$

From Hamilton's action principle (7), (9) and (10) follows the Euler-Lagrange equation



$$\frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{\partial \mathcal{L}}{\partial \left(\frac{dx^\mu}{d\lambda}\right)} + \frac{\partial \mathcal{L}}{\partial \left(\frac{d^2 x^\mu}{d\lambda^2}\right)} - \dots = 0 \quad (11)$$

If  $\mathcal{L}$  contains second or higher order derivatives then the order of this differential equation may be of an order higher than two. In order to implement the common assumption that the proper initial conditions at  $\lambda = \lambda_0$  for a particle are its 'position'  $x^\mu(\lambda_0)$  and its 'velocity'  $\frac{dx^\mu}{d\lambda}(\lambda_0)$  we want (11) to be a second order differential equation and thus we assume that  $\mathcal{L}$  does not contain second or higher order derivatives:

$$\mathcal{L} = \mathcal{L}\left(x^\mu, \frac{dx^\mu}{d\lambda}\right) \quad (12)$$

Next we want to find an action for a free particle. For a free particle all points of spacetime appear to be equivalent. Hence we want the action of a free particle to be invariant under translations in  $\mathbf{M}$ :

$$S[x^\mu + a^\mu] = S[x^\mu] \quad (13)$$

This is satisfied if

$$S_{\lambda_1 \lambda_2}[x^\mu] = \int_{\lambda_1}^{\lambda_2} \mathcal{L}\left(\frac{dx^\mu}{d\lambda}\right) d\lambda \quad (14)$$

Since all Lorentz coordinate systems are equivalent one assumes that the action is a scalar under Lorentz transformations. All Lorentz scalars which can be made out of the tangent vector  $\dot{\gamma}$  are functions of  $\eta(\dot{\gamma}, \dot{\gamma}) = \eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}$ . Hence

$$S_{\lambda_1 \lambda_2}[x^\mu] = \int_{\lambda_1}^{\lambda_2} f\left(\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}\right) d\lambda \quad (15)$$

The choice of the parametrization of the worldline has no physical relevance. The requirement of invariance under reparametrizations of  $\ell$  gives, e.g.

$$S_{\lambda_1 \lambda_2}[x^\mu] = \alpha \int_{\lambda_1}^{\lambda_2} \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda \quad (16)$$

where  $\alpha$  is a constant. Defining the Lagrangian  $L$  by

$$S = \int L dt \quad (17)$$

where  $t$  is the time coordinate, we find

$$L = \alpha \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} = \alpha c \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} \quad (18)$$

where  $\mathbf{v} = \left(\frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt}\right)$  is the velocity of the particle. In the non-relativistic regime  $\mathbf{v}^2 \ll c^2$  (18) gives

$$L = \alpha c - \alpha \frac{\mathbf{v}^2}{2c} + \dots \quad (19)$$

In the second term in the right-hand side of (19) we recognize the non-relativistic kinetic energy  $\frac{1}{2}m\mathbf{v}^2$ , with the constant  $m$  the mass of the particle. Hence we put  $\alpha = mc$  and the action of a free massive particle (for the part of its worldline between the events  $\gamma(\lambda_1)$  and  $\gamma(\lambda_2)$ ) becomes

$$S = -mc \int_{\lambda_1}^{\lambda_2} \sqrt{\eta(\dot{\gamma}, \dot{\gamma})} d\lambda \quad (20)$$

The action  $S$  is proportional to the arclength between  $\gamma(\lambda_1)$  and  $\gamma(\lambda_2)$ . For the action (20) Hamilton's action principle gives, via its Euler–Lagrange equations, rise to

$$\frac{d^2 x^\mu}{ds^2} = 0 \quad (21)$$

Hence worldlines of free massive particles are straight lines in Minkowski space-time. This is up to expectations, since a free particle is not subjected to accelerations.

The action (20) is not suitable for massless particles ( $m = 0$ ). There exists a nice alternative to the action (20). It has the advantage that it does not contain a square root, and moreover, it can also be used for massless particles. It requires, however, the introduction of an auxiliary field on the worldline  $\ell$ . This auxiliary field is taken to be a so-called one-bein field  $e$  on the one-dimensional manifold  $\ell$ .

More generally one defines an  $n$ -bein field on an  $n$ -dimensional semi-Riemannian manifold  $(M, g)$  [the metric tensor  $g$  has signature  $(1, -1, \dots, -1)$ ] to be a set of  $n$  contravariant vector fields  $\{e_a \mid a = 0, \dots, n-1\}$  on  $M$  such that for all  $e_a(x) \in T_x(M)$  one has the orthonormality relations

$$g(e_a(x), e_b(x)) = \eta_{ab} \quad (a, b = 0, \dots, n-1) \quad (22)$$

where  $\eta_{00} = -\eta_{11} = -\eta_{22} = \dots = 1$  and  $\eta_{ab} = 0$  for  $a \neq b$ . The (holonomic) components  $e_a^\mu$  of an  $n$ -bein field  $e_a$  are introduced as usual by

$$e_a = e_a^\mu \partial_\mu \quad (23)$$

where  $\partial_\mu$  is a tangent vector to the  $\mu$ -th coordinate curve. From (22) follows for the components of the  $e_a$ 's

$$g_{\mu\nu} e_a^\mu(x^\sigma) e_b^\nu(x^\sigma) = \eta_{ab} \quad (24)$$

and this implies

$$g_{\mu\nu} = \eta^{ab} e_{a\mu}(x^\sigma) e_{b\nu}(x^\sigma) = \eta_{ab} e_\mu^a(x^\sigma) e_\nu^b(x^\sigma) \quad (25)$$

where  $(\eta^{ab})$  is the inverse matrix of  $(\eta_{ab})$  (i.e.  $\eta^{ab} = \eta_{ab}$ ),  $e_{a\mu} := g_{\mu\nu}e_a^\nu$  and  $e_\mu^a(x^\sigma) := \eta^{ab}e_{b\mu}(x^\sigma)$ . From (25) one sees that the metric tensor is uniquely determined by the  $n$ -bein fields, and so one can use  $n$   $n$ -bein fields  $e_a$  ( $a = 0, 1, \dots, n-1$ ) instead of the metric tensor field  $g$ . To a given metric  $g$  their correspond, however, several  $n$ -bein fields. Indeed, let the linear transformation

$$\Lambda(x): v \in T_x(M) \mapsto \hat{v} = \Lambda(x)v \in T_x(M) \quad (26)$$

be a Lorentz transformation, i.e. for all  $v, w \in T_x(M)$  one has  $g(v, w) = g(\hat{v}, \hat{w})$ , then the vectors  $\{\hat{e}_a(x) \mid a = 0, 1, \dots, n-1\}$  with  $\hat{e}_a(x) := \Lambda(x)e_a(x)$  form an  $n$ -bein if  $\{e_a(x) \mid a = 0, 1, \dots, n-1\}$  is an  $n$ -bein. One has  $\hat{e}_a(x) = \Lambda(x)e_a(x) = e_b(x)\Lambda(x)^b_a$  and  $g(\hat{e}_a(x), \hat{e}_b(x)) = g(e_a(x), e_b(x)) = \eta_{ab}$  and consequently

$$\eta_{cd}\Lambda(x)^c_a\Lambda(x)^d_b = \eta_{ab} \quad (27)$$

and we see that  $(\Lambda(x)^a_b)$  is a Lorentz matrix. The  $n$ -bein fields  $e_a(x)$  and  $\hat{e}_a(x)$  give rise to the same metric

$$g_{\mu\nu} = \eta^{ab}e_{a\mu}(x^\sigma)e_{b\nu}(x^\sigma) = \eta^{ab}\hat{e}_{a\mu}(x^\sigma)\hat{e}_{b\nu}(x^\sigma) \quad (28)$$

For the invariant volume element  $\sqrt{|g|}d^n x$  with  $g := \det(g_{\mu\nu})$  one has  $\sqrt{|g|}d^n x = e d^n x$  where  $e := \det(e_{a\mu})$ . Notice that under a coordinate transformation  $(x^\mu) \mapsto (x^{\mu'})$  one has

$$e_{a\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} e_{a\mu} \quad (29)$$

We are now in the position to introduce an alternative action  $S = S[x^\mu, e]$  of a particle in Minkowski spacetime. This new action depends on the worldline  $\ell$  of the particle and on an auxiliary field  $e$  on  $\ell$ . This action has to be a Lorentz scalar and independent of its parametrization  $x^\mu = x^\mu(\lambda) = \check{x}^\mu(\check{\lambda})$  where  $\check{\lambda} = \check{\lambda}(\lambda)$  is considered as a coordinate transformation on  $\ell$ . Since

$$\frac{d\check{x}^\mu}{d\check{\lambda}} = \frac{d\lambda}{d\check{\lambda}} \frac{dx^\mu}{d\lambda} \quad (30)$$

and [see (29)]

$$\check{e} = \frac{d\lambda}{d\check{\lambda}} e \quad (31)$$

one has

$$\check{e}^{-1} \frac{d\check{x}^\mu}{d\check{\lambda}} = e^{-1} \frac{dx^\mu}{d\lambda} \quad (32)$$

The reparametrization invariant volume element of  $\ell$  is  $e d\lambda$ . In view of all this the alternative action is taken to be

$$S = -\frac{c}{2} \int_{\lambda_1}^{\lambda_2} \left( e^{-2} \eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} + m^2 \right) e d\lambda \quad (33)$$

Clearly this action does not contain a square root and the constant  $m$  can be taken zero in it. Its Euler–Lagrange equations for  $x^\mu$  and  $e$  respectively read

$$\left. \frac{d}{d\epsilon} S[x^\mu + \epsilon y^\mu, e] \right|_{\epsilon=0} = 0 \quad (34)$$

and

$$\left. \frac{d}{d\epsilon} S[x^\mu, e + \epsilon f] \right|_{\epsilon=0} = 0 \quad (35)$$

for all  $y^\mu = y^\mu(\lambda)$  and  $f = f(\lambda)$  vanishing at the endpoints having parameters  $\lambda_1$  and  $\lambda_2$ . From (35) follows

$$-\frac{c}{2} \frac{\partial(e^{-1}\eta(\dot{\gamma}, \dot{\gamma}) + m^2 e)}{\partial e} = 0 \quad (36)$$

or

$$e = \frac{\sqrt{\eta(\dot{\gamma}, \dot{\gamma})}}{m} \quad (37)$$

For  $m \neq 0$  insertion of (37) in (33) gives back the original action (20). For  $m = 0$  (35) gives

$$\eta(\dot{\gamma}, \dot{\gamma}) = 0 \quad (38)$$

i.e. the tangent vectors to the worldline of a massless particle are lightlike.

From (34) one obtains

$$\frac{d}{d\lambda} \left( e^{-2} \frac{dx^\mu}{d\lambda} \right) = 0 \quad (39)$$

For  $m \neq 0$  one can choose  $\lambda = s$  (the arclength) and then  $e = m^{-1}$  and (39) becomes

$$\frac{d^2}{ds^2} x^\mu = 0 \quad (40)$$

For a massless particle ( $m = 0$ ) one cannot choose  $\lambda$  to be equal to the arclength. However  $\lambda$  can be chosen such that  $e = \text{constant}$  and (39) then gives

$$\frac{d^2}{d\lambda^2} x^\mu = 0 \quad (41)$$

*Classical bosonic string* Whereas at a fixed time  $t$  a particle is represented by a point (event) in spacetime, a classical bosonic string is (at a fixed time  $t$ ) by definition a differentiable curve  $\alpha_t: \xi \in [a, b] \subset \mathbf{R} \mapsto \alpha_t(\xi) \in \mathbf{E}_t^{n-1}$ , where  $\mathbf{E}_t^{n-1} := \{x \in \mathbf{M} \mid x^0 = ct\}$  is the physical space at time  $t$ . For the sake of convenience we assume that this curve is not selfintersecting. There are two cases to distinguish. Either this curve has two different endpoints, i.e.

$\alpha_t(a) \neq \alpha_t(b)$ , and then the string is called open, or the endpoints coincide, i.e.  $\alpha_t(a) = \alpha_t(b)$ , and then the string is called closed.

The history of the point  $\alpha_t(\xi_0)$  ( $\xi_0$  fixed) of the string gives rise to a worldline in  $\ell_{\xi_0} \in \mathbf{M}$ . Hence, the evolution of a string in the course of time gives rise to a subset  $\Sigma := \cup_{\xi} \ell_{\xi} \subset \mathbf{M}$  consisting of a one-parameter family (parameter  $\xi$ ) of worldlines. The subset  $\Sigma$  is the so-called the worldsheet of the string and it is the range of the map

$$(\xi, t) \in [a, b] \times \mathbf{R} \mapsto (ct, \alpha_t(\xi)) \in \mathbf{M} \quad (42)$$

We assume that the worldsheet  $\Sigma$  is a two-dimensional submanifold of  $\mathbf{M}$ . For an open string this submanifold has boundaries, formed by the worldlines  $\ell_a$  and  $\ell_b$  of the endpoints of the string. The worldsheet of a closed string has a tubelike form. Similar to the situation prevailing for a particle, the history of a string is determined by the points of the worldsheet  $\Sigma$  and its parametrization is irrelevant in this respect. Consequently we will not restrict to the parameters  $(\xi, t)$  of (42), but use more general parameters  $(\sigma, \tau)$  instead. Let us furthermore assume for the sake of convenience that they coordinatize the whole worldsheet  $\Sigma$ . The coordinate expression of the worldsheet then reads

$$x^{\mu} = x^{\mu}(\sigma, \tau) \quad (\mu = 0, 1, \dots, n-1) \quad (43)$$

where the real parameters  $\sigma$  and  $\tau$  are taken to run through the intervals  $0 \leq \sigma \leq \pi$  and  $-\infty < \tau < \infty$  for an open string and they are taken to run through the intervals  $0 \leq \sigma \leq 2\pi$  and  $-\infty < \tau < \infty$  for a closed string. Furthermore we assume that  $\tau$  is a timelike parameter, i.e., the tangent vectors to the curves on the worldsheet with  $\sigma = \text{constant}$ ,  $-\infty < \tau < \infty$  are timelike, i.e.

$$\eta_{\mu\nu} \frac{\partial x^{\mu}}{\partial \tau} \frac{\partial x^{\nu}}{\partial \tau} > 0 \quad (44)$$

and that  $\sigma$  is a spacelike parameter, i.e., the tangent vectors to the curves on the worldsheet with  $\tau = \text{constant}$ ,  $0 \leq \sigma \leq \pi$  are spacelike, i.e.

$$\eta_{\mu\nu} \frac{\partial x^{\mu}}{\partial \sigma} \frac{\partial x^{\nu}}{\partial \sigma} < 0 \quad (45)$$

For a closed string one has the condition

$$x^{\mu}(0, \tau) = x^{\mu}(2\pi, \tau) \quad (-\infty < \tau < \infty) \quad (46)$$

Instead of requiring  $0 \leq \sigma \leq 2\pi$  it is sometimes more convenient to extend the range of  $\sigma$  to all of  $\mathbf{R}$  and make thereafter the identification

$$x^{\mu}(\sigma, \tau) = x^{\mu}(\sigma + 2\pi, \tau) \quad (-\infty < \sigma, \tau < \infty) \quad (47)$$

The dynamics of the classical string is also described by means of Hamilton's action principle. As its dynamical variables are taken the functions  $x^{\mu} = x^{\mu}(\sigma, \tau)$  [see (43)]. The action  $S$  of the string is then a real-valued functional of these functions  $x^0 = x^0(\sigma, \tau), \dots, x^{n-1} = x^{n-1}(\sigma, \tau)$ :

$$S: (x^0, \dots, x^{n-1}) \mapsto S = S[x^\mu] \in \mathbf{R} \quad (48)$$

It has the form

$$S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{a\pi} d\sigma \mathcal{L}(x^\mu, \partial_\sigma x^\mu, \partial_\tau x^\mu) \quad (49)$$

where  $a = 1$  for an open string and  $a = 2$  for a closed string. Hamilton's action principle states that in the set of all a priori possible histories (worldsheets  $\Sigma$ ) the action  $S$  is stationary for the actually happening history of the string, or more explicitly:

$$\left. \frac{d}{d\lambda} S[x^\mu + \lambda y^\mu] \right|_{\lambda=0} = 0 \quad (50)$$

for all functions  $y^\mu = y^\mu(\sigma, \tau)$  satisfying

$$y^\mu(\sigma, \tau_i) = y^\mu(\sigma, \tau_f) = 0 \quad (0 \leq \sigma \leq a\pi) \quad (51)$$

Indeed, one has [see (49)]

$$\left. \frac{d}{d\lambda} S[x^\mu + \lambda y^\mu] \right|_{\lambda=0} = \int_{\tau_i}^{\tau_f} d\tau \int_0^{a\pi} d\sigma \left( \frac{\partial \mathcal{L}}{\partial x^\mu} y^\mu + \frac{\partial \mathcal{L}}{\partial x^{\mu'}} y^{\mu'} + \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \dot{y}^\mu \right) \Big|_{\lambda=0} \quad (52)$$

where

$$\dot{x}^\mu := \frac{\partial x^\mu}{\partial \tau}, \quad x^{\mu'} := \frac{\partial x^\mu}{\partial \sigma} \quad (53)$$

Hence

$$\begin{aligned} \left. \frac{d}{d\lambda} S[x^\mu + \lambda y^\mu] \right|_{\lambda=0} &= \int_{\tau_i}^{\tau_f} d\tau \int_0^{a\pi} d\sigma \left( \frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{\partial}{\partial \sigma} \frac{\partial \mathcal{L}}{\partial x^{\mu'}} - \frac{\partial}{\partial \tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \right) y^\mu \\ &+ \left. \frac{\partial \mathcal{L}}{\partial x^{\mu'}} y^\mu \right|_{\sigma=0}^{\sigma=a\pi} + \left. \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} y^\mu \right|_{\tau=\tau_i}^{\tau=\tau_f} \end{aligned} \quad (54)$$

Hence the action  $S$  is stationary if the right-hand side of (54) vanishes. The last term in the right-hand side of (54) vanishes due to (51), i.e. we determine the evolution of the string  $x^\mu = x^\mu(\sigma, \tau)$  for a given initial configuration  $x^\mu = x^\mu(\sigma, \tau_i)$  and a given final configuration  $x^\mu = x^\mu(\sigma, \tau_f)$ . Thus we find

$$\int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \left( \frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{\partial}{\partial \sigma} \frac{\partial \mathcal{L}}{\partial x^{\mu'}} - \frac{\partial}{\partial \tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \right) y^\mu + \left. \frac{\partial \mathcal{L}}{\partial x^{\mu'}} y^\mu \right|_{\sigma=0}^{\sigma=a\pi} = 0 \quad (55)$$

Recall that the factor

$$\pi_\mu \equiv \pi^\tau{}_\mu := \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \quad (56)$$

appearing in (54) is called the conjugate momentum. Here we have introduced the more elaborate notation  $\pi^\tau{}_\mu$  since we similarly define

$$\pi^\sigma{}_\mu := \frac{\partial \mathcal{L}}{\partial x^{\mu'}} = \frac{\partial \mathcal{L}}{\partial (\partial_\sigma x^\mu)} \quad (57)$$

For all functions  $y^\mu = y^\mu(\sigma, \tau)$  with a support contained in  $0 < \sigma < a\pi$  and  $\tau_i < \tau < \tau_f$  one has

$$\int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \left( \frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{\partial}{\partial \sigma} \frac{\partial \mathcal{L}}{\partial x^{\mu'}} - \frac{\partial}{\partial \tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \right) y^\mu = 0 \quad (58)$$

since for these functions the last term in the left-hand side of (55) vanishes. Hence

$$\frac{\partial \mathcal{L}}{\partial x^\mu} - \partial_\tau \frac{\partial \mathcal{L}}{\partial (\partial_\tau x^\mu)} - \partial_\sigma \frac{\partial \mathcal{L}}{\partial (\partial_\sigma x^\mu)} = 0 \quad (59)$$

for  $0 < \sigma < a\pi$  and  $\tau_i < \tau < \tau_f$ . The equations (59) are called the Euler-Lagrange equations of the string. An alternative form of these equations reads

$$\frac{\partial \mathcal{L}}{\partial x^\mu} - \partial_\tau \pi^\tau{}_\mu - \partial_\sigma \pi^\sigma{}_\mu = 0 \quad (60)$$

For a closed string the last term in the left-hand side of (55) vanishes due to  $y^\mu(0, \tau) = y^\mu(2\pi, \tau)$  and (46) [supplied with similar conditions for its derivatives with respect to  $\dot{x}^\mu$  and  $x^{\mu'}$ ]. For an open string (55) gives, using (59),

$$\frac{\partial \mathcal{L}}{\partial x^{\mu'}} \Big|_{\sigma=0} y^\mu(0) - \frac{\partial \mathcal{L}}{\partial x^{\mu'}} \Big|_{\sigma=a\pi} y^\mu(\pi) = 0 \quad (61)$$

Since  $y^\mu(0)$  and  $y^\mu(\pi)$  are independent we arrive for an open string at the boundary conditions

$$\pi^\sigma{}_\mu \Big|_{\sigma=0} = \frac{\partial \mathcal{L}}{\partial x^{\mu'}} \Big|_{\sigma=0} = 0, \quad \pi^\sigma{}_\mu \Big|_{\sigma=\pi} = \frac{\partial \mathcal{L}}{\partial x^{\mu'}} \Big|_{\sigma=\pi} = 0 \quad (62)$$

One candidate for the action  $S$  of the string is the so-called Nambu-Goto action. This action is proportional to the area of the worldsheet of the string between  $\tau = \tau_i$  and  $\tau = \tau_f$ . This choice is similar to the action of a particle, which is taken to be proportional to the arclength between two events of its worldline. Analogously the area of a surface does not change if this surface is reparametrized. Recall that the area  $d\mathcal{A}$  of an infinitesimal element of the worldsheet bounded by the four curves  $\tau = \text{constant}$ ,  $\tau + d\tau = \text{constant}$ ,  $\sigma = \text{constant}$  and  $\sigma + d\sigma = \text{constant}$

$$d\mathcal{A} = \sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2} d\sigma d\tau \quad (63)$$

The Nambu–Goto action is taken to be

$$S = -\frac{T}{c} \int_{\tau_i}^{\tau_f} d\tau \int_0^{a\pi} d\sigma \sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2} \quad (64)$$

where the real number  $T$  is called the string tension. Observe that  $\dot{x}^\mu$  and  $x'^\mu$  are timelike and spacelike tangent vectors to the worldsheet of the string, respectively. The Lagrangian density corresponding to (64) reads

$$\mathcal{L} = -\frac{T}{c} \sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2} \quad (65)$$

The Euler–Lagrange equation (59) reads for this Lagrangian density

$$\frac{\partial}{\partial \tau} \left\{ \frac{x'^\mu (\dot{x} \cdot x') - \dot{x}^\mu \dot{x}^2}{\sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2}} \right\} + \frac{\partial}{\partial \sigma} \left\{ \frac{\dot{x}^\mu (\dot{x} \cdot x') - x'^\mu \dot{x}^2}{\sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2}} \right\} = 0 \quad (66)$$

This equation of motion is a non-linear partial differential equation. The first term in curly brackets in the left hand of (66) is proportional to the conjugate momentum  $\pi_\mu \equiv \pi^\tau{}_\mu = \pi_\mu(\sigma, \tau)$  [see (56)]. The latter reads for the Nambu–Goto action

$$\pi^\tau{}_\mu = -\frac{T}{c} \frac{x'^\mu (\dot{x} \cdot x') - \dot{x}^\mu \dot{x}^2}{\sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2}} \quad (67)$$

Furthermore

$$\pi^\sigma{}_\mu = -\frac{T}{c} \frac{\dot{x}^\mu (\dot{x} \cdot x') - x'^\mu \dot{x}^2}{\sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2}} \quad (68)$$

From (67) and (68) one gets the following identities

$$\pi^\tau \cdot x' = 0, \quad (\pi^\tau)^2 + \frac{T^2}{c^2} x'^2 = 0 \quad (69)$$

$$\pi^\sigma \cdot \dot{x} = 0, \quad (\pi^\sigma)^2 + \frac{T^2}{c^2} \dot{x}^2 = 0 \quad (70)$$

For an open string the boundary conditions at the endpoints read [see (62)]

$$\pi^{\sigma\mu} \Big|_{\sigma=0} = \frac{\dot{x}^\mu (\dot{x} \cdot x') - x'^\mu \dot{x}^2}{\sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2}} \Big|_{\sigma=0} = 0 \quad (71)$$

$$\pi^{\sigma\mu} \Big|_{\sigma=\pi} = \frac{\dot{x}^\mu (\dot{x} \cdot x') - x'^\mu \dot{x}^2}{\sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2}} \Big|_{\sigma=\pi} = 0 \quad (72)$$



From (70), (71) and (72) follows

$$\dot{x}^2|_{\sigma=0} = \dot{x}^2|_{\sigma=\pi} = 0 \quad (73)$$

This means that the endpoints of an open string move with the speed of light. For a closed string (66) is supplemented by the boundary condition (46) or (47) a closed string.

By means of a reparametrization

$$\sigma \mapsto \tilde{\sigma} = \tilde{\sigma}(\sigma, \tau), \quad \tau \mapsto \tilde{\tau} = \tilde{\tau}(\sigma, \tau) \quad (74)$$

one can simplify the Euler–Lagrange equation (66). A reparametrization induces also a new coordinate expression  $\tilde{x}^\mu = \tilde{x}^\mu(\sigma, \tau)$  of the worldsheet, where  $\tilde{x}^\mu(\tilde{\sigma}, \tilde{\tau}) := x^\mu(\sigma, \tau)$ . Recall that the evolution of the string is completely described by the events (points) on its worldsheet  $\Sigma$  and the choice of the parametrization of the worldsheet has no observable consequences. It can be shown that the reparametrization can be chosen in such a way that  $\tilde{x}$  satisfies the constraints

$$\dot{\tilde{x}}^2 + \tilde{x}'^2 = 0, \quad \dot{\tilde{x}} \cdot \tilde{x}' = 0 \quad (75)$$

Equivalently one has  $\dot{\tilde{x}}^2 \pm 2\dot{\tilde{x}} \cdot \tilde{x}' + \tilde{x}'^2 = 0$ , or

$$(\dot{\tilde{x}} \pm \tilde{x}')^2 = 0 \quad (76)$$

where the tilde has been and will be suppressed from now on. A parametrization of the worldsheet which satisfies (75) or (76) is called the orthonormal gauge. In this gauge the Euler–Lagrange equation (66) reads

$$\ddot{x} - x'' = 0 \quad (77)$$

This equation has the form of the wave equation in two dimensions. Hence in the orthonormal gauge the equation of motion (77) turns out to be linear. However, the constraints (75) [and (76)] on  $x^\mu$  are non-linear. In the orthonormal gauge the conjugate momentum (67) becomes

$$\pi^\mu = \frac{T}{c} \dot{x}^\mu \quad (78)$$

The boundary conditions (71) and (72) for an open string take the simple form

$$x'^\mu(\sigma, \tau)|_{\sigma=0} = x'^\mu(\sigma, \tau)|_{\sigma=\pi} = 0 \quad (79)$$

The general solution of the equation of motion (77) is now given by means of a Fourier series with respect to the variable  $\sigma$ . For an open string this solution, also satisfying the boundary conditions (79), reads

$$\frac{1}{\ell} x^\mu(\sigma, \tau) = \sum_{n=0}^{\infty} c_n^\mu(\tau) \cos n\sigma \quad (80)$$

where (77) implies that the  $c_n^\mu(\tau)$ 's satisfy

$$\ddot{c}_n^\mu + n^2 c_n^\mu = 0 \quad (n = 0, 1, 2, \dots) \quad (81)$$

and  $\ell$  is a real constant, to be fixed later in such a way that it sets, in a convenient manner, the scale of the integration constants resulting from the integration of (81). The general solution of (81) reads for  $n = 0$

$$c_0^\mu(\tau) = q^\mu + p^\mu \tau \quad (82)$$

and for the other values of  $n$  one has

$$c_n^\mu(\tau) = q_n^\mu \cos n\tau + \dot{q}_n^\mu \frac{\sin n\tau}{n} \quad (n = 1, 2, \dots) \quad (83)$$

where  $q^\mu = c_0^\mu(0)$ ,  $p^\mu = \dot{c}_0^\mu(0)$ ,  $q_n^\mu = c_n^\mu(0)$  and  $\dot{q}_n^\mu = \dot{c}_n^\mu(0)$  are real integration constants. Insertion of (82) and (83) in (80) gives

$$\frac{1}{\ell} x^\mu(\sigma, \tau) = q^\mu + p^\mu \tau + \sum_{n=1}^{\infty} \left\{ q_n^\mu \cos n\tau + \dot{q}_n^\mu \frac{\sin n\tau}{n} \right\} \cos n\sigma \quad (84)$$

Introduction of so-called harmonic oscillator variables by

$$\alpha_{\pm n}^\mu := \frac{1}{2} (\dot{q}_n^\mu \pm i n q_n^\mu) \quad (n = 1, 2, \dots) \quad (85)$$

permits us to rewrite (84) as

$$\begin{aligned} \frac{1}{\ell} x^\mu(\sigma, \tau) &= q^\mu + p^\mu \tau + i \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau} \cos n\sigma \\ &= q^\mu + p^\mu \tau + \frac{i}{2} \sum_{n \neq 0} \left\{ \frac{\alpha_n^\mu}{n} e^{-in(\tau+\sigma)} + \frac{\alpha_n^\mu}{n} e^{-in(\tau-\sigma)} \right\} \end{aligned} \quad (86)$$

For future reference we note that (85) gives for the complex conjugate

$$\alpha_n^{\mu*} = \alpha_{-n}^\mu \quad (87)$$

From (78) and (86) follows

$$\begin{aligned} \frac{c}{T\ell} \pi_\mu &= p^\mu + \sum_{n \neq 0} \alpha_n^\mu e^{-in\tau} \cos n\sigma \\ &= p^\mu + \frac{1}{2} \sum_{n \neq 0} \left\{ \alpha_n^\mu e^{-in(\tau+\sigma)} + \alpha_n^\mu e^{-in(\tau-\sigma)} \right\} \end{aligned} \quad (88)$$

Setting

$$\alpha_0^\mu := p^\mu \quad (89)$$

this can be rewritten as

$$\frac{c}{T\ell}\pi_\mu = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left\{ \alpha_n^\mu e^{-in(\tau+\sigma)} + \alpha_n^\mu e^{-in(\tau-\sigma)} \right\} \quad (90)$$

Next we indicate, following section 1.5 of [10], how the constraints (76) can be expressed in harmonic oscillator variables. From (86) follows, using (89),

$$\frac{1}{\ell}(\dot{x}^\mu(\sigma, \tau) \pm x^{\mu'}(\sigma, \tau)) = \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in(\tau \pm \sigma)} \quad (91)$$

Both these results are related by interchanging  $\sigma$  and  $-\sigma$ . Hence, if we declare (86) to hold on the extended interval  $-\pi \leq \sigma \leq \pi$ , instead of only on the interval  $0 \leq \sigma \leq \pi$ , we can combine the two constraints (76) into one. Namely

$$\{\dot{x}(\sigma, \tau) + x'(\sigma, \tau)\}^2 = 0, \quad (-\pi \leq \sigma \leq \pi) \quad (92)$$

where [see (91)]

$$\frac{1}{\ell}(\dot{x}^\mu(\sigma, \tau) + x^{\mu'}(\sigma, \tau)) = \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in(\tau+\sigma)}, \quad (-\pi \leq \sigma \leq \pi) \quad (93)$$

From (92) and (93) we get

$$\begin{aligned} \frac{1}{\ell^2}(\dot{x}(\sigma, \tau) + x'(\sigma, \tau))^2 &= \sum_{m,p=-\infty}^{\infty} \alpha_m^\mu \alpha_{p\mu} e^{-i(m+p)(\tau+\sigma)} \\ &= \sum_{n=-\infty}^{\infty} e^{-in(\tau+\sigma)} \sum_{m=-\infty}^{\infty} \alpha_m \alpha_{n-m} \end{aligned} \quad (94)$$

Hence the constraints (76) can be expressed as

$$L_n = 0 \quad (n \in \mathbf{Z}) \quad (95)$$

where  $L_n$  is defined by

$$L_n := \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_m \alpha_{n-m} \quad (96)$$

Similar to (33) there exists an alternative to the Nambu–Goto action which depends on  $x^\mu = x^\mu(\sigma, \tau)$  and the two-bein  $e_a [e_a(\sigma, \tau) \in T_x(\Sigma)]$  where  $x$  is the point in the worldsheet  $\Sigma$  with coordinates  $(\sigma, \tau)$ . This action is usually called the Polyakov action [9] although it can already be found in [1] by L. Brink, P. Di Vecchia and P. Howe. Let us denote the coordinates in the worldsheet  $\Sigma$  also by  $(\xi^\alpha) \equiv (\xi^0, \xi^1) := (\sigma, \tau)$ . Recall [see (22)] that one has the orthonormality relations

$$g(e_a(\sigma, \tau), e_b(\sigma, \tau)) = \eta_{ab} \quad (a, b \in \{0, 1\}) \quad (97)$$

where  $\eta_{00} = -\eta_{11} = 1$  and  $\eta_{01} = \eta_{10} = 0$ . Consequently

$$g_{\alpha\beta} = \eta^{ab} e_{a\alpha} e_{b\beta} = \eta_{ab} e_a^\alpha e_b^\beta \quad (\alpha, \beta, a, b \in \{0, 1\}) \quad (98)$$

where  $(\eta^{ab})$  is the inverse matrix of  $(\eta_{ab})$ , i.e.  $\eta^{ab} = \eta_{ab}$  and  $e_{a\alpha} := g_{\alpha\beta} e_a^\beta$  [compare (25)]. From this one sees that the components of the 2-dimensional metric tensor  $g$  are uniquely determined by the 2-bein fields. The invariant volume element of the worldsheet reads  $ed\sigma d\tau$  where  $e := \det(e_a^\beta) = \det(\sqrt{-g_{\alpha\beta}})$ . A simple reparametrization invariant scalar containing the string variable  $x^\mu$  reads  $g^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} = e_a^\alpha e_b^\beta \eta^{ab} \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu}$ . In view of this the Polyakov action is taken to be

$$S = \frac{T}{2c} \int_{\tau_i}^{\tau_f} d\tau \int_0^{a\pi} d\sigma e e_a^\alpha e_b^\beta \eta^{ab} \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} \quad (99)$$

Since  $g^{\alpha\beta} = e_a^\alpha e_b^\beta \eta^{ab} \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu}$  and  $e = \sqrt{-g}$  [ $g := \det(g_{\alpha\beta})$ ] this can be rewritten as

$$S = \frac{-T}{2c} \int_{\tau_i}^{\tau_f} d\tau \int_0^{a\pi} d\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} \quad (100)$$

From

$$\frac{d}{d\lambda} S[x^\mu, g^{\alpha\beta} + \lambda k^{\alpha\beta}] \Big|_{\lambda=0} = 0 \quad (101)$$

one finds

$$\partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} = \frac{1}{2} g_{\alpha\beta} (g^{\gamma\delta} \partial_\gamma x^\mu \partial_\delta x^\nu) \quad (102)$$

Setting

$$h_{\alpha\beta} := \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} \quad (103)$$

or as matrix equation

$$(h_{\alpha\beta}) = \begin{pmatrix} \dot{x}^2 & \dot{x} \cdot x' \\ \dot{x} \cdot x' & x'^2 \end{pmatrix} \quad (104)$$

we get [see (102)]

$$h_{\alpha\beta} = \frac{1}{2} g_{\alpha\beta} (g^{\gamma\delta} \partial_\gamma x^\mu \partial_\delta x^\nu) \quad (105)$$

Hence

$$\det(h_{\alpha\beta}) = \frac{1}{4} (g^{\gamma\delta} \partial_\gamma x^\mu \partial_\delta x^\nu)^2 \det(g_{\alpha\beta}) \quad (106)$$

or [compare (104)]

$$\sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2} = \sqrt{-\det(h_{\alpha\beta})} = \frac{1}{2} (g^{\gamma\delta} \partial_\gamma x^\mu \partial_\delta x^\nu) \sqrt{-g} \quad (107)$$

From this consequence of the equation of motion one sees that the Nambu action (64) and the Polyakov action (99) give rise to the same equation of motion of the string variable  $x^\mu = x^\mu(\sigma, \tau)$ . The following obvious remark has to be made. The equivalence of the Nambu action and the Polyakov action is here only derived in a classical setting. In a quantum mechanical theory this has to be reconsidered.

*Quantization* We shortly recall the elements of quantum mechanics which are needed here. In quantum theory the fundamental classical dynamical variables are replaced by linear operators acting on a Hilbert space  $\mathcal{H}$ . To begin with these operators are sufficiently characterized by means of (anti)commutation relations. A state of the system is represented by a non-zero vector in  $\mathcal{H}$ , called a state vector. In Dirac's notation vectors in  $\mathcal{H}$  are denoted by  $|\psi\rangle, |\phi\rangle, \dots$  and the inner product of the vectors  $|\psi\rangle$  and  $|\phi\rangle$  is denoted by  $\langle\psi|\phi\rangle$ . The vectors  $|\psi\rangle \neq 0$  and  $\lambda|\psi\rangle$  ( $\lambda \in \mathbb{C}, \lambda \neq 0$ ) represent the same state of the system. Observables (measurable physical quantities) are represented by self-adjoint operators acting on  $\mathcal{H}$ . These self-adjoint operators are thought to be constructed from the operators representing the fundamental dynamical variables. The relation between the quantum theoretical description of states and observables on the one hand and the outcome of experiments on the other hand is found in the probability interpretation. This can be stated as follows. Let the state of a system be represented by a non-zero vector  $|\psi\rangle \in \mathcal{H}$ . One measures the observable  $\mathcal{A}$ , represented by the self-adjoint operator  $A$ , repeatedly. For everyone of these measurements the system which is prepared in the same state  $|\psi\rangle$ . The outcome of a measurement is always a spectral value of the self-adjoint operator  $A$ . The mean value (expectation value)  $\langle A \rangle$  of all these measurements is given by

$$\langle A \rangle = \frac{\langle\psi|A|\psi\rangle}{\langle\psi|\psi\rangle} \quad (108)$$

Observe that the left-hand expression contains the quantity which is to be determined experimentally, and the right-hand expression contains the quantity which is to be calculated theoretically.

There are several ways to quantize a classical system. For the classical string theory with the Nambu–Goto action we outline here the so-called old covariant approach. In this approach, which is similar to the Gupta–Bleuler formalism of quantum electrodynamics, the classical canonical variables  $x^\mu(\sigma, \tau)$  and  $\pi^\mu(\sigma, \tau)$  are replaced by linear operators, and the latter will also be denoted by  $x^\mu(\sigma, \tau)$  and  $\pi^\mu(\sigma, \tau)$ . Due to this replacement the classical variables  $q^\mu, p^\mu$  and  $\alpha_n^\mu$  in (86) also become linear operators and (86) becomes an operator identity. The equal-time canonical commutation relations of the operators  $x^\mu(\sigma, \tau)$  and  $\pi^\mu(\sigma, \tau)$  read

$$[x^\mu(\sigma, \tau), \pi_\nu(\sigma', \tau)] = i\hbar\delta(\sigma - \sigma')\delta_\nu^\mu \quad (109)$$

$$[x^\mu(\sigma, \tau), x^\nu(\sigma', \tau)] = 0 \quad (110)$$

and

$$[\pi^\mu(\sigma, \tau), \pi^\nu(\sigma', \tau)] = 0 \quad (111)$$

Observe that (109) can be rewritten as

$$[\pi^\mu(\sigma, \tau), x^\nu(\sigma', \tau)] = i\hbar\delta(\sigma - \sigma')\eta^{\mu\nu} \quad (112)$$

where  $\eta^{\mu\nu} := \eta_{\mu\nu}$ . These commutation relations lead to commutation relations for the operators  $q^\mu$ ,  $p^\mu$  and  $\alpha_n^\mu$ . Choosing

$$\ell := \sqrt{\frac{c\hbar}{\pi T}} \quad (113)$$

one obtains

$$[p^\mu, q^\nu] = i\eta^{\mu\nu} \quad (114)$$

$$[\alpha_m^\mu, \alpha_n^\nu] = n\delta_{m, -n}\eta^{\mu\nu} \quad (115)$$

Equation (86) has to be replaced by

$$\alpha_n^{\mu\dagger} := \alpha_{-n}^\mu \quad (116)$$

where  $\dagger$  means hermitian conjugation. From (115) and (116) follows

$$[\alpha_m^\mu, \alpha_n^{\nu\dagger}] = -n\delta_{mn}\eta^{\mu\nu} \quad (m, n = 1, 2, \dots) \quad (117)$$

The next step will be the construction of the state vector space of the bosonic string. The theory of the linear harmonic oscillator will serve as a stepping stone.

*Linear harmonic oscillator* The operators  $\{\alpha_n^\mu\}$  are, indeed, with good reason called harmonic oscillator variables as can be seen from the following results pertinent to the harmonic oscillator in one dimension. We give here a rather elaborate discussion of the energy eigenvalue problem of the harmonic oscillator since we will encounter a similar situation in the treatment of the state vector space of the bosonic string.

The hamiltonian  $H$  of the one-dimensional harmonic oscillator is given by

$$\begin{aligned} H &= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \\ &= \frac{\hbar\omega}{2} \left( \frac{p^2}{m\hbar\omega} + \frac{m\omega q^2}{\hbar} \right) \end{aligned} \quad (118)$$

where  $p$  is the momentum operator and  $q$  the position operator of the particle oscillating with frequency  $\nu = \omega/2\pi$ . The operators  $p$  and  $q$  are characterized by their commutation relation

$$[p, q] = \frac{\hbar}{i} \quad (119)$$

Introducing the dimensionless operators

$$P := \sqrt{\frac{1}{m\hbar\omega}}p, \quad Q := \sqrt{\frac{m\omega}{\hbar}}q \quad (120)$$

one gets

$$H = \frac{\hbar\omega}{2} (Q^2 + P^2) \quad (121)$$

and

$$[P, Q] = \frac{\hbar}{i} \quad (122)$$

Hence, inspired by the identity  $a^2 + b^2 = (a - ib)(a + ib)$  for c-numbers, one can write for the operators  $P$  and  $Q$

$$Q^2 + P^2 = (Q - iP)(Q + iP) + i[P, Q] = a^\dagger a + 1 \quad (123)$$

where

$$a := \frac{Q + iP}{\sqrt{2}}, \quad a^\dagger := \frac{Q - iP}{\sqrt{2}} \quad (124)$$

From (121) and (123) follows

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) \quad (125)$$

Due to (122) and (124) the operators  $a^\dagger$  and  $a$  satisfy the commutation relations

$$[a, a^\dagger] = 1 \quad (126)$$

Immediate consequences of (125) and (126) are

$$[H, a^\dagger] = \hbar\omega a^\dagger \quad (127)$$

and

$$[H, a] = -\hbar\omega a \quad (128)$$

From (127) and (128) follows that  $a$  and  $a^\dagger$  can be used to produce new eigenvectors of  $H$  from old ones. Indeed, if  $|\psi\rangle$  is an eigenvector of  $H$  with eigenvalue  $E$ , i.e.  $H|\psi\rangle = E|\psi\rangle$ , then (128) gives via action on  $|\psi\rangle$  that  $H(a|\psi\rangle) - E(a|\psi\rangle) = -\hbar\omega(a|\psi\rangle)$  and consequently  $a|\psi\rangle$  is either an eigenvector of  $H$  with eigenvalue  $E - \hbar\omega$  or it is the zero vector. Similarly, if  $|\psi\rangle$  is an eigenvector of  $H$  with eigenvalue  $E$  then  $a^\dagger|\psi\rangle$  is either an eigenvector of  $H$  with eigenvalue  $E + \hbar\omega$  or it is the zero vector. Instead of the hamiltonian  $H$  one can equally well analyze the so-called number operator

$$N := a^\dagger a \quad (129)$$

From [see (125) and (129)]

$$H = \hbar\omega(N + \frac{1}{2}) \quad (130)$$

one sees that  $H$  and  $N$  have the same set of eigenvectors. Denoting the eigenvalue equation of  $N$  by

$$N|n\rangle = n|n\rangle \quad (131)$$

where we have labelled an eigenvector by its eigenvalue  $n \in \mathbf{R}$ . The analogues of (127) and (128) are

$$[N, a^\dagger] = a^\dagger \quad (132)$$

and

$$[N, a] = -a \quad (133)$$

From (133) it follows that  $N(a|n\rangle) - n(a|n\rangle) = -(a|n\rangle)$  and consequently  $a|n\rangle$  is either an eigenvector of  $N$  with eigenvalue  $n - 1$ , i.e.  $a|n\rangle \propto |n - 1\rangle$ , or it is the zero vector. Similarly, it follows from (132) that  $a^\dagger|n\rangle$  is either an eigenvector of  $N$  with eigenvalue  $n + 1$ , i.e.  $a^\dagger|n\rangle \propto |n + 1\rangle$ , or it is the zero vector. It is assumed that the eigenvalues of  $N$  (and for that matter of  $H$ ) are non-degenerate. Note that

$$H|n\rangle = \hbar\omega(n + \frac{1}{2})|n\rangle \quad (134)$$

Furthermore the eigenvectors are taken to be normalized

$$\langle n|n\rangle = 1 \quad (135)$$

The spectrum of  $N$  is a subset of the non-negative real numbers since for an eigenvalue  $n$  of  $N$  one has

$$0 \leq \|a|n\rangle\|^2 = \langle n|a^\dagger a|n\rangle = \langle n|N|n\rangle = n \quad (136)$$

[see (129) and (131)]. Hence, if  $n \neq 0$  then  $a|n\rangle \neq 0$  is an eigenvector of  $N$  with eigenvalue  $n - 1$  or using (135), (136) and choosing a suitable phase factor in the eigenvectors one has

$$a|n\rangle = \sqrt{n}|n - 1\rangle \quad (n > 0) \quad (137)$$

We are now ready to conclude that the eigenvalues of  $N$  are  $n = 0, 1, 2, \dots$ . Suppose that  $|x\rangle$  ( $x > 0$ ) is an eigenvector of  $N$  with  $x \neq 0, 1, 2, \dots$ . Let  $p$  be a natural number and  $p > x$ , then  $a^p|x\rangle$  is an eigenvector of  $N$  [see (137)] with a negative eigenvalue, namely the eigenvalue  $x - p < 0$ . This is a contradiction and hence it follows that the only possible eigenvalues of  $N$  are given by  $n = 0, 1, 2, \dots$ . Let  $|n_0\rangle$  be an eigenvector of  $N$  then one gets by repeated action of  $a$  the new eigenvectors  $a|n_0\rangle \propto |n_0 - 1\rangle, a|n_0 - 1\rangle \propto |n_0 - 2\rangle, \dots, a|2\rangle \propto |1\rangle, a|1\rangle \propto |0\rangle$ . At this point this ladder has to stop, since otherwise one would get a negative eigenvalue. Thus [compare (136)]



$$a|0\rangle = 0 \quad (138)$$

Observe in this formula the difference between the unit vector  $|0\rangle$  and the zero vector in its right-hand side. By repeated action of  $a^\dagger$  one obtains the new eigenvectors  $a^\dagger|n_0\rangle \propto |n_0 + 1\rangle, a^\dagger|n_0 + 1\rangle \propto |n_0 + 2\rangle, \dots$ . This part of the ladder does not stop [compare (136)]. Hence for each  $n = 0, 1, 2, \dots$  there is an eigenvector  $|n\rangle$  and the action of  $a$  on the eigenvectors of this ladder is given by (137) and (138). The eigenvalue  $E_n$  of  $|n\rangle$  for the hamiltonian  $H$  is given by [see (134)]

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \quad (n = 0, 1, 2, \dots) \quad (139)$$

The eigenvector  $|0\rangle$  has the lowest energy eigenvalue and the corresponding state is called the ground state of the harmonic oscillator. Under the action of  $a$  on eigenvectors of  $H$  the energy eigenvalue decreases by an ‘energy quantum’  $\hbar\omega$  and therefore  $a$  is called the annihilation operator. From  $a^\dagger|n\rangle \propto |n + 1\rangle$  follows by means of (135) and a proper choice of the phase factors in the normalized eigenvectors

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad (n = 0, 1, 2, \dots) \quad (140)$$

Hence, under the action of  $a^\dagger$  on eigenvectors of  $H$  the energy eigenvalue increases by an ‘energy quantum’  $\hbar\omega$  and therefore  $a^\dagger$  is called the creation operator. From (140) follows

$$|n\rangle = \frac{1}{\sqrt{n}}a^\dagger|n-1\rangle = \frac{1}{\sqrt{n(n-1)}}a^{\dagger 2}|n-2\rangle = \dots \quad (141)$$

or

$$|n\rangle := \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \quad (142)$$

In summary, the state vector space  $\mathcal{H}$  of the linear harmonic oscillator is spanned by an orthonormal basis  $\{|n\rangle \mid n = 0, 1, 2, \dots\}$  of eigenvectors of  $N$  and  $H$  with eigenvalues  $n$  and  $\hbar\omega(n + \frac{1}{2})$ , respectively. The ground state vector has the lowest eigenvalue and it is characterized by  $a|0\rangle = 0$  [see (138)]. All other eigenvectors  $|n\rangle$  of  $N$  and  $H$  are obtained from the ground state vector  $|0\rangle$  by means of the repeated action of the creation operator [see (141)]. The energy eigenvalues are given by (139). We conclude this section with a comment on the role played by the following modification in the sign in the right-hand of (126):

$$[a, a^\dagger] = \pm 1 \quad (143)$$

when we still adhere to  $a|0\rangle = 0$ . Let

$$|\psi\rangle := a^\dagger|0\rangle \quad (144)$$

then the scalar product of this vector with itself is given by

$$\langle \psi | \psi \rangle = \langle 0 | a a^\dagger | 0 \rangle = \langle 0 | a^\dagger a \pm 1 | 0 \rangle \quad (145)$$

or

$$\langle \psi | \psi \rangle = \pm 1 \quad (146)$$

Hence with a minus sign in the right-hand side of (143) the vector space resulting by a repeated action of  $a^\dagger$  on  $|0\rangle$  has an indefinite scalar product.

*State vector space of the bosonic string* Inspired by all this we next turn to the construction of the state vector space  $\mathcal{H}$  of the bosonic string. This is the Hilbert space containing all vectors representing physically realizable states. The action of the operators  $q^\mu$ ,  $p^\mu$  and  $\alpha_n^\mu$  has to be defined on a dense domain in  $\mathcal{H}$ . Below we will see that the state vector space  $\mathcal{H}$  is a proper subspace of the representation space  $\mathcal{R}$  of the operators  $q^\mu$ ,  $p^\mu$  and  $\alpha_n^\mu$ . Since the observables  $x^\mu(\sigma, \tau)$  and  $\pi^\mu(\sigma, \tau)$  are represented by self-adjoint operators, the operators  $q^\mu$  and  $p^\mu$  are self-adjoint operators. The operators  $\alpha_n^\mu$  ( $n = \pm 1, \pm 2, \dots$ ) are not self-adjoint. By comparison of (117) and (126) one sees that  $\alpha_n^{\mu\dagger}$  ( $n = 1, 2, \dots$ ) are creation operators, and that  $\alpha_n^\mu$  ( $n = 1, 2, \dots$ ) are annihilation operators. The analogue of the ground state of the harmonic oscillator is here the vector denoted by  $|0\rangle \in \mathcal{R}$  (same notation as for the harmonic oscillator!), however, it is now characterized by

$$\alpha_n^\mu |0\rangle = 0 \quad (n = 1, 2, \dots) \quad (147)$$

$$p^\mu |0\rangle = 0 \quad (148)$$

Let  $\gamma \equiv (\gamma^\mu) \in \mathbf{R}^n$  and  $q \equiv (q^\mu) = (q^0, \dots, q^{n-1})$  then the vector

$$|\gamma\rangle := e^{-i\gamma \cdot q} |0\rangle \quad (149)$$

is an eigenvector of the pairwise commuting operators  $p \equiv (p^\mu) = (p^0, \dots, p^{n-1})$ :

$$p^\mu |\gamma\rangle = \gamma^\mu |\gamma\rangle \quad (150)$$

This is an immediate consequence of (148), (114) and

$$[p^\mu, e^{-i\gamma \cdot q}] = \gamma^\mu e^{-i\gamma \cdot q} \quad (151)$$

Observe that

$$\langle \gamma | \gamma' \rangle = 0 \quad \text{for } \gamma \neq \gamma' \quad (152)$$

since eigenvectors of a self-adjoint operator with different eigenvalues are orthogonal. From the vectors  $|\gamma\rangle$  one obtains a basis of  $\mathcal{R}$  by means of repeated action of creation operators  $\alpha_n^{\mu\dagger}$  ( $n = 1, 2, \dots$ ) on these vectors. In this way we get the basis

$$\begin{aligned}
& |\gamma\rangle, \alpha_{n_1}^{\mu_1 \dagger} |\gamma\rangle, \alpha_{n_1}^{\mu_1 \dagger} \alpha_{n_2}^{\mu_2 \dagger} |\gamma\rangle, \dots, \\
& \alpha_{n_1}^{\mu_1 \dagger} \alpha_{n_2}^{\mu_2 \dagger} \dots \alpha_{n_p}^{\mu_p \dagger} |\gamma\rangle, \dots
\end{aligned} \tag{153}$$

where  $n_1, n_2, \dots, n_p, \dots = -1, -2, \dots$  ( $p \in \mathbf{N}_+$ ).

Next we implement the constraints (76) in the quantum mechanical setting. Replacement of the  $\alpha_n$ 's in (96) by their corresponding operators leads to ambiguous results, since the classical quantities  $\alpha_n$  and  $\alpha_{-n}$  commute, whereas their corresponding operators do not commute. This is usually remedied by replacing products of classical quantities by the so-called normal ordered product of their corresponding operators. Normal ordering of a product of creation and/or annihilation operators means that in this product these operators are put in an ordering where all annihilation operators occur to the left of all creation operators. Note that all creation operators commute, and likewise, that all annihilation operators commute. Normal ordering of a product of creation and/or annihilation operators is indicated by putting colons around this product, i.e.

$$\begin{aligned}
& : \alpha_{-3} \alpha_7 : := \alpha_{-3} \alpha_7 \\
& : \alpha_7 \alpha_{-3} : := \alpha_{-3} \alpha_7
\end{aligned} \tag{154}$$

Thus the operator corresponding to the classical  $L_n$  is defined to be

$$L_n := \frac{1}{2} \sum_{m=-\infty}^{\infty} : \alpha_m \cdot \alpha_{n-m} : \tag{155}$$

where  $\alpha_n \equiv (\alpha_n^\mu)$ . This gives in particular

$$L_0 := \frac{1}{2} p^2 + \sum_{n>0} \alpha_n^\dagger \cdot \alpha_n = \frac{1}{2} p^2 + N \tag{156}$$

where we introduced

$$N := \sum_{n>0} \alpha_n^\dagger \cdot \alpha_n \tag{157}$$

Since the classical constraints  $L_n = 0$  cannot be implemented as operator equations  $L_n = 0$  one proceeds as follows. One defines a so-called physical state  $|\psi\rangle$  by the conditions

$$L_n |\psi\rangle = 0 \quad (n > 0) \tag{158}$$

and

$$L_0 |\psi\rangle = \lambda |\psi\rangle \quad (\lambda \in \mathbf{R}) \tag{159}$$

These states form the subspace of physical states. From (158) follows

$$\langle \psi | L_{-n} = \langle \psi | L_n^\dagger = 0 \quad (n > 0) \quad (160)$$

This [(158)—(160)] gives rise to

$$\langle \psi | L_n - \delta_{0n} \lambda | \psi \rangle = 0 \quad (161)$$

for all  $n \in \mathbf{Z}$  and physical states  $|\psi\rangle$ . This is the quantum mechanical analogue of the classical constraints  $L_n = 0$ . In the subsequent section we will argue that the dimension of Minkowski spacetime  $n = 26$  and  $\lambda = 1$ . Restricted to the subspace of physical states one has

$$\frac{1}{2}p^2 + N = 1 \quad (162)$$

The momentum of the string is given by

$$P^\mu = \int_0^\pi \pi^\mu(\sigma) d\sigma = \sqrt{\frac{\pi T \hbar}{c}} p^\mu \quad (163)$$

With the definition of the operator of the square of the mass  $M^2 := P^2$  one gets on the subspace of physical states using (162)

$$M^2 = \frac{2\pi \hbar T}{c} (N - 1) \quad (164)$$

*Tachyon* From (150) and (156) follows

$$-\frac{1}{2}\gamma^2 |\gamma\rangle = -\frac{1}{2}p^2 |\gamma\rangle = L_0 |\gamma\rangle = |\gamma\rangle \quad (165)$$

Hence the mass-squared of the states (149), i.e. the eigenvalue  $m^2$  of  $M^2$ , is given by

$$m^2 = \gamma^2 = -2 \quad (166)$$

Hence the momentum  $P^\mu$  of these states is spacelike, whereas for particles with a speed not exceeding the speed of light the momentum  $P^\mu$  is timelike or lightlike. The states (149) describe a particle with a superluminal speed. Such particles are called tachyons. They are not observed in nature. By introducing additional fermionic degrees of freedom in the string model the tachyon can be eliminated from the theory. The resulting string model is called a superstring. Here we will, however, stick to the bosonic string. Actually we will consider the closed bosonic string, because it is this string for which we have the Frenkel-Kac-Segal mechanism.

*Closed string* The parameter expression of the worldsheet of the closed string is given by  $x^\mu = x^\mu(\sigma, \tau)$  where  $0 \leq \sigma \leq 2\pi$ . Actually one can take  $-\infty < \sigma < \infty$  if one supplements this by

$$x^\mu(\sigma, \tau) = x^\mu(\sigma + 2\pi, \tau) \quad (167)$$

The general solution of the Euler–Lagrange equation in the orthonormal gauge (77)

$$x^\mu(\sigma, \tau) = g^\mu(\tau + \sigma) + h^\mu(\tau - \sigma) \quad (168)$$

where  $g$  and  $h$  are arbitrary (twice-differentiable) functions. The solutions  $x^\mu(\sigma, \tau) = g^\mu(\tau + \sigma)$  are called left-movers and the solutions  $x^\mu(\sigma, \tau) = h^\mu(\tau - \sigma)$  are called right-movers. We will rewrite (168) as

$$\frac{1}{\ell} x^\mu(\sigma, \tau) = \frac{1}{2} x_L^\mu(e^{i(\tau+\sigma)}) + \frac{1}{2} x_R^\mu(e^{i(\tau-\sigma)}) \quad (169)$$

Furthermore

$$x_L^\mu(z) = q^\mu + p_L^\mu(\tau + \sigma) + i \sum_{n \neq 0} \frac{\alpha_n^{L\mu}}{n} z^{-n} \quad (z := e^{i(\tau+\sigma)}) \quad (170)$$

and

$$x_R^\mu(z) = q^\mu + p_R^\mu(\tau - \sigma) + i \sum_{n \neq 0} \frac{\alpha_n^{R\mu}}{n} z^{-n} \quad (z := e^{i(\tau-\sigma)}) \quad (171)$$

From (167) and (169)—(171) follows

$$p_L^\mu = p_R^\mu \equiv p \quad (172)$$

The canonical commutation relations of  $x_L^\mu$ ,  $x_R^\mu$ ,  $\pi_\mu^L$  and  $\pi_\mu^R$  lead to

$$[\alpha_m^{L\mu}, \alpha_n^{L\nu}] = [\alpha_m^{R\mu}, \alpha_n^{R\nu}] = m \delta_{m,-n} \eta^{\mu\nu} \quad (173)$$

$$[\alpha_m^{L\mu}, \alpha_n^{R\nu}] = 0 \quad (174)$$

$$[q^\mu, p^\nu] = \frac{i}{2} \eta^{\mu\nu}, \quad [q^\mu, q^\nu] = [p^\mu, p^\nu] = 0 \quad (175)$$

Defining

$$L_n^L := \frac{1}{2} \sum_m : \alpha_{m\mu}^L \alpha_{n-m}^{L\mu} :, \quad L_n^R := \frac{1}{2} \sum_m : \alpha_{m\mu}^R \alpha_{n-m}^{R\mu} : \quad (176)$$

the states  $|\psi\rangle$  of the physical subspace are characterized by

$$L_n^L |\psi\rangle = 0, \quad L_n^R |\psi\rangle = 0 \quad (n = 1, 2, \dots) \quad (177)$$

and

$$(L_0^L + L_0^R) |\psi\rangle = 2\psi \quad (178)$$

where again the dimension of Minkowski spacetime is taken to be 26. The generator of translations in  $\sigma$  is  $L_n^L - L_n^R$ . Since the choice of the origin  $\sigma = 0$  is arbitrary this generator has to be zero:

$$L_0^L = L_0^R \quad (179)$$

The momentum of the closed string is given by

$$P^\mu = 2\sqrt{\frac{\pi T \hbar}{c}} p^\mu \quad (180)$$

With  $M^2 := P^2$  one finds

$$M^2 = \frac{4\pi \hbar T}{c} (N_L + N_R - 21) \quad (181)$$

where

$$N_L := \sum_{n>0} \alpha_n^{L\dagger} \cdot \alpha_n^L \quad N_R := \sum_{n>0} \alpha_n^{R\dagger} \cdot \alpha_n^R \quad (182)$$

From (176), (179) and (182) one obtains

$$N_L = N_R \quad (183)$$

### 3. ANOMALIES

Field theories occur in two species: a classical version and a quantum theoretical version. The quantum theoretical version is mostly obtained from the classical version by a so-called quantization procedure. Some aspects of the classical theory have their analogues in the corresponding quantum theory. In many classical field theories symmetries and conserved quantities give rise to symmetries and conserved quantities in the corresponding quantum theory. There are, however, exceptions to this rule. In such a situation one speaks of anomalies. For the classical version we will consider here a so-called classical lagrangian field theory.

*Classical lagrangian field theory* The main ingredients of such a theory are its fundamental dynamical variables, called fields, its action and its observables (measurable quantities). The fields are a finite set of maps

$$\phi_i: x \in M \mapsto \phi_i(x) \in \mathbf{C} \quad (i = 1, \dots, N) \quad (184)$$

where  $M$  is a differentiable manifold. Usually  $M$  is the spacetime manifold consisting of all possible point-events. An example is provided by the four-dimensional Minkowski space  $(\mathbf{M}, \eta)$  of special relativity and more generally the four-dimensional semi-Riemannian manifold  $(M, g)$  of general relativity. Here  $g$  is a metric on  $M$  with signature  $(1, -1, -1, -1)$ . An example of a field on spacetime  $M$  is the electromagnetic field strength, which is represented by a 2-form  $F$  on  $M$ . However,  $M$  may as well be the two-dimensional worldsheet  $\Sigma$  of string theory. The fundamental dynamical variables are in this case the maps

$$\Sigma \rightarrow \mathbf{M} \quad (185)$$

with coordinate expressions  $x^\mu = x^\mu(\sigma, \tau)$ . Classical mechanics can also be considered as a field theory. Then  $M$  is the one-dimensional manifold of time  $\mathbf{T}$ . In the lagrangian formalism field are then maps

$$t \in \mathbf{T} \mapsto (q(t), \dot{q}(t)) \in T(Q) \quad (186)$$

where  $T(Q)$  is the tangent bundle of the configuration space  $Q$ .

Fields will collectively be denoted by  $\Phi = (x^1, \dots, x^N)$ . The action  $S$  is a given real functional of the fields, i.e. a map

$$S: \Phi \mapsto S[\Phi] \in \mathbf{R} \quad (187)$$

It is assumed that the action has the form

$$S = \int_{\Omega} \mathcal{L}(\Phi, \partial_\mu \Phi, \partial_\mu \partial_\nu \Phi, \dots) d^n x \quad (188)$$

where  $\Omega$  is an open subset of  $M$ . For the sake of convenience we have assumed here that it is contained in the coordinate neighbourhood of a chart  $(U, \kappa)$  of the differentiable manifold  $M$ . Denoting the coordinates of  $x \in U$  by  $\kappa(x) = (x^1, \dots, x^n)$  we have  $\partial_\mu = \frac{\partial}{\partial x^\mu}$ ,  $d^n x := dx^1 dx^2 \dots dx^n$  and  $\mu, \nu = 1, \dots, n$ . The function real function  $\mathcal{L} = \mathcal{L}(\Phi, \partial_\mu \Phi, \partial_\mu \partial_\nu \Phi, \dots)$  is called the lagrangian (density). The dependence of  $S$  on  $\Omega$  will be made explicit by the notation  $S \equiv S_\Omega$ .

The equations of motion of the fields are obtained from the action by means of Hamilton's action principle. For the particular case of the string this was already given in (50). Its general form reads

$$\left. \frac{d}{d\lambda} S_\Omega[\Phi + \lambda \Psi] \right|_{\lambda=0} = 0 \quad (189)$$

for all fields  $\Psi$  which vanish on the boundary  $\partial\Omega$  of  $\Omega$ . The Euler–Lagrange equations corresponding to (189) read for  $x \in \Omega$

$$\frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^i)} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu \phi^i)} - \dots = 0 \quad (i = 1, \dots, n) \quad (190)$$

In many cases one has a lagrangian  $\mathcal{L} = \mathcal{L}(\Phi, \partial_\mu \Phi)$  and then only the first two terms in the left-hand side appear.

The observables of a classical lagrangian field theory are functions  $\mathcal{O} = \mathcal{O}(\phi^1, \dots, \phi^n)$  of the fields. Some observables are related via Noether's theorem to smooth families of symmetries of the action. A transformation of the fields

$$\Phi \mapsto \hat{\Phi} = G(\Phi) \quad (191)$$

is called a symmetry transformation if there are functions  $\mathcal{M}^\kappa$  of  $\Phi, \partial_\mu \Phi, \dots$  such that

$$\mathcal{L}(\hat{\Phi}, \partial_\mu \hat{\Phi}, \dots) = \mathcal{L}(\Phi, \partial_\mu \Phi, \dots) + \partial_\kappa \mathcal{M}^\kappa(\Phi, \partial_\mu \Phi, \dots) \quad (192)$$

This implies, using Gauss' theorem,

$$S[\hat{\Phi}] = S[\Phi] + \int_{\partial\Omega} \Lambda^\kappa(\Phi, \partial_\mu \Phi, \dots) d\sigma_\kappa \quad (193)$$

where  $d\sigma_\kappa$  is a surface element of the boundary  $\partial\Omega$  of  $\Omega$ . From (193) and Hamilton's action principle [see (189)] follows that  $\hat{\Phi}$  is a solution of the Euler–Lagrange equations iff  $\Phi$  is a solution of the Euler–Lagrange equations.

We now state Noether's theorem. Let

$$\hat{\Phi} = G(\Phi, \varepsilon) = \Phi + \varepsilon F(\Phi) + O(\varepsilon^2) \quad (194)$$

that is

$$\hat{\phi}^i = G^i(\Phi, \varepsilon) = \phi^i + \varepsilon F^i(\Phi) + O(\varepsilon^2) \quad (i = 1, \dots, N) \quad (195)$$

be a smooth one-parameter family of symmetry transformations. Then (192) takes the form

$$\left. \frac{d}{d\varepsilon} \mathcal{L}(\hat{\Phi}, \partial_\mu \hat{\Phi}, \dots) \right|_{\varepsilon=0} =: \partial_\kappa \Lambda^\kappa(\Phi, \partial_\mu \Phi, \dots) \quad (196)$$

On the other hand

$$\left. \frac{d}{d\varepsilon} \mathcal{L}(\hat{\Phi}, \partial_\mu \hat{\Phi}, \dots) \right|_{\varepsilon=0} = \sum_{i=1}^N \left( \frac{\partial \mathcal{L}}{\partial \phi^i} F^i + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^i)} \partial_\mu F^i \right) \quad (197)$$

and using the Euler–Lagrange equations for the first factor in the first term under the summation sign in the right-hand side of (197) one gets

$$\left. \frac{d}{d\varepsilon} \mathcal{L}(\hat{\Phi}, \partial_\mu \hat{\Phi}, \dots) \right|_{\varepsilon=0} = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^i)} F^i \right) \quad (198)$$

From (196) and (198) we get the following continuity equation (differential conservation equation):

$$\partial_\mu J^\mu = 0 \quad (199)$$

where the so-called Noether current  $J^\mu$  is defined by

$$J^\mu := \left( \sum_{i=1}^N \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^i)} F^i \right) - \Lambda^\mu \quad (200)$$

We will now indicate how (199) leads to global conservation laws. Firstly (199) implies

$$0 = \int_\Omega \partial_\mu J^\mu d^n x = \int_{\partial\Omega} J^\mu d\sigma_\mu \quad (201)$$

Secondly, let the boundary  $\partial\Omega$  consist out of two disjoint connected subsets  $\Sigma_1$  and  $\Sigma_2$  and let  $\partial\Sigma$  be the common boundary of the closures of  $\Sigma_1$  and  $\Sigma_2$ . Reversing the orientation of e.g.  $\Sigma_2$  permits us to rewrite (201) as



$$\int_{\Sigma_1} J^\mu d\sigma_\mu = \int_{\Sigma_2} J^\mu d\sigma_\mu \quad (202)$$

Defining

$$Q(\Sigma) := \int_{\Sigma} J^\mu d\sigma_\mu \quad (203)$$

we conclude that  $Q(\Sigma)$  is the same for all  $\Sigma$ 's with the same boundary.

The meaning of all this becomes clear when we consider in the setting of special relativity the example of the electromagnetic four-current  $j^\mu(x^\nu) = (c\rho(ct, \mathbf{x}), \mathbf{j}(ct, \mathbf{x}))$  where  $\rho$  is the electric charge density and  $\mathbf{j} = (j^1, j^2, j^3)$  is the electric current density ( $c$  is speed of light). We assume that  $j^\mu$  vanishes sufficiently rapidly at spatial infinity  $|\mathbf{x}| \rightarrow \infty$ . The charge at time  $t$  is defined by the spatial integral

$$Q(t) := \int_{\Sigma(t)} \rho(ct, \mathbf{x}) d\mathbf{x} \quad (204)$$

with  $\Sigma(t)$  the hyperplane in Minkowski spacetime consisting of all events with  $x^0 = ct = \text{constant}$  with respect to a given Lorentz coordinate system. Here we have the pendant of (199):

$$0 = \partial_\mu j^\mu = \frac{\partial \rho}{\partial t} + \nabla \mathbf{j} \quad (205)$$

The independence of the hyperplane  $\Sigma(t)$  [see (202)] leads in this case to charge conservation

$$\frac{dQ(t)}{dt} = 0 \quad (206)$$

Similarly  $Q(\Sigma)$  from (203) gives rise to a conservation law for a suitable one-parameter family (parameter  $\tau$ , e.g.  $\tau = t$ ) of hypersurfaces  $\Sigma = \Sigma(\tau)$ . The following fact is very important: if translations  $x^\mu \mapsto x^\mu + \varepsilon^\mu$  in Minkowski spacetime give rise to symmetry transformations then Noether's theorem leads to the conservation law of energy and momentum. Indeed, spacetime translations give rise to the following 4-parameter family of transformations of fields [compare (195)]

$$\hat{\phi}^i(x^\mu) := \phi^i(x^\mu + \varepsilon^\mu) = \phi^i(x^\mu) + \varepsilon^\kappa \partial_\kappa \phi^i(x^\mu) + O(\varepsilon^2) \quad (i = 1, \dots, N) \quad (207)$$

If the lagrangian does not depend explicitly on  $(x)$  then

$$\begin{aligned} \left. \frac{\partial}{\partial \varepsilon^\kappa} \mathcal{L}(\phi^i(x^\mu + \varepsilon^\mu), \partial_\nu \phi^i(x^\mu + \varepsilon^\mu), \dots) \right|_{\varepsilon=0} &= \\ \frac{\partial}{\partial x^\kappa} \mathcal{L}(\phi^i(x^\mu), \partial_\nu \phi^i(x^\mu), \dots) &= \partial_\mu (\delta_\kappa^\mu \mathcal{L}) \end{aligned} \quad (208)$$

Hence in this case the transformations (207) are symmetry transformations. Let us denote the Noether current  $J^\mu$  corresponding to the one-parameter family obtained from (207) by setting  $\varepsilon^\mu = 0$  for  $\mu \neq \kappa$  by  $\Theta_\kappa^\mu$  then

$$\Theta_\kappa^\mu = \left( \sum_{i=1}^N \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \partial_\kappa \phi^i \right) - \delta_\kappa^\mu \mathcal{L} \quad (209)$$

and one has

$$\partial_\mu \Theta_\kappa^\mu = 0 \quad (210)$$

The conserved current  $\Theta_\kappa^\mu$  is called the canonical energy–momentum tensor. The four conserved charges

$$P_\kappa := \int \Theta_\kappa^0 dx \quad (211)$$

form the four–momentum of the system. In particular  $cP_0 = H$  is the energy of the system and

$$\Theta_0^0 = \left( \sum_{i=1}^N \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^i)} \partial_0 \phi^i \right) - \mathcal{L} \quad (212)$$

is the energy density. Notice that these observables are indeed functions of the fields.

*Feynman quantization* There are various quantization procedures for classical theories. Here we will use Feynman’s quantization procedure although it is in many instances mathematically ill-founded due to its liberal use and handling of functional integrals. In this method all kinds of physical quantities are expressed in terms of functionals. We will restrict ourselves here to the so-called  $\tau$ –fuctions (also somewhat misleadingly called Green’s functions). The  $\tau$ –fuctions contain all the information about the scattering of the particles of the quantum field theory in question. Scattering of particles is described in quantum theory by means of scattering amplitudes. Scattering amplitudes are complex valued functions of the variables which are used to describe the states of the incoming and the outgoing particles of a scattering process. The probability that a particular scattering process occurs is obtained from the square of the absolute value of the corresponding scattering amplitude. So the description of scattering processes is completely encoded in their scattering amplitudes. In turn, the scattering amplitudes can be obtained by means of the reduction formulas of Lehmann, Symanzik and Zimmerman from the above mentioned  $\tau$ –fuctions.

We will define  $\tau$ –fuctions and their expression in terms of Feynman’s functional integrals starting from the classical field theory with one real scalar field  $\phi$  on Minkowski spacetime. The lagrangian is taken to be

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \mathcal{V}(\phi) \quad (213)$$

with e.g.

$$\mathcal{V} = \frac{1}{3!}\phi^3 + \frac{1}{4!}\phi^4 \quad (214)$$

The corresponding hamiltonian density  $\mathcal{H}$  is obtained by a Legendre transformation, i.e. let

$$\pi := \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad (\dot{\phi} \equiv \frac{\partial \phi}{\partial t}) \quad (215)$$

be the conjugate momentum then under the assumption that  $\dot{\phi}$  can be solved from (215), giving  $\dot{\phi} = f(\phi, \pi)$ , then

$$\mathcal{H} := \left( \pi \dot{\phi} - \mathcal{L} \right) \Big|_{\dot{\phi}=f(\phi,\pi)} \quad (216)$$

Observe that

$$\mathcal{H} := \Theta_0^0 \quad (217)$$

For the lagrangian (213) one has (setting  $c = 1$ )

$$\pi = \partial_0 \phi \quad (218)$$

and (216) yields

$$\mathcal{H} := \frac{1}{2}\pi^2 + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{1}{2}m^2\phi^2 + \mathcal{V}(\phi) \quad (219)$$

and integration over a hyperplane  $\Sigma(t)$  gives the hamiltonian of the classical theory

$$\begin{aligned} H &:= \int \mathcal{H} dx \\ &= \int \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{1}{2}m^2\phi^2 + \mathcal{V}(\phi) \right\} dx \end{aligned} \quad (220)$$

In the quantization procedure the real scalar fields  $\phi(x)$  and  $\pi(x)$  are replaced by a self-adjoint operator fields denoted by  $\hat{\phi}(x)$  and  $\hat{\pi}(x)$ . The Hamilton operator  $\underline{H}$  is obtained from (220) by replacing the classical fields  $\phi(x)$  and  $\pi(x)$  by their corresponding operator fields  $\hat{\phi}(x)$  and  $\hat{\pi}(x)$  and specifying some ordering prescription, e.g. normal ordering in the interaction picture (in the interaction picture the fields are free fields). The spectrum of  $H$  is bounded from below and it is assumed that there has been added such a constant to the hamiltonian that its lowest eigenvalue is equal to zero. The eigenstate with the lowest eigenvalue of  $H$  is called the ground state or more specifically the vacuum state. For  $m^2 > 0$  it can be made easily plausible that the ground state is non-degenerate and an normalized vector representing this state is denoted by  $|0\rangle$ . Hence

$$H|0\rangle = 0 \quad \langle 0|0\rangle = 1 \quad (221)$$

We are now finally in the position that we are able to define the  $\tau$ -fuctions:

$$\tau(x_1, x_2, \dots, x_n) := \langle 0|T(\underline{\phi}(x_1)\underline{\phi}(x_2)\dots\underline{\phi}(x_n))|0\rangle \quad (222)$$

wherein  $T$  denotes the time-ordering instruction defined by

$$T(\underline{\phi}(x_1)\underline{\phi}(x_2)\dots\underline{\phi}(x_n)) := \underline{\phi}(x_{i_1})\underline{\phi}(x_{i_2})\dots\underline{\phi}(x_{i_n}) \quad (223)$$

if  $x_{i_1}^0 > x_{i_2}^0 > \dots > x_{i_n}^0$ . Hence the time-ordering instruction puts the operators in an ordering such every operator with an earlier time  $x^0$  stands to the right of each operator with a later time. In Feynman's quantization a  $\tau$ -function (like other physical quantities) is expressed in a functional integral. A functional integral is the (mystifying) modification of an ordinary integral  $\int f(x)dx$  obtained by replacing the integration variable  $x$  by a function (in our case the scalar fields on Minkowski spacetime  $\mathbf{M}$ ) and the the function  $f = f(x)$  by a complex-valued functional  $F = F[\phi]$ . Notationally this has the following effect

$$\int dx f(x) \rightarrow \int \mathcal{D}\phi F[\phi] \quad (224)$$

A rigorous definition of a functional integral exists only for special simple cases. In the physicist's approach functional integrals are often introduced by means of a lattice approximation. Take for instance a cubic lattice in some hypercube  $\mathcal{C}$  in Minkowski spacetime  $\mathbf{M}$ . Let us denote the lattice points by  $x_1, x_2, \dots, x_N$ . We assume that there exists some approximation  $\phi^{(a)}$  of the classical field  $\phi$  which is completely determined by the values of the field  $\phi$  in the lattice points  $x_1, x_2, \dots, x_N$ , more in particular

$$\phi^{(a)}(x_i) = \phi(x_i) \equiv \phi_i \quad (i = 1, \dots, N) \quad (225)$$

Herewith a scalar field  $\phi$  is replaced by a finite set of real numbers  $\phi_1, \dots, \phi_N$ . The value of the functional  $F[\phi]$  is now approximated by  $F[\phi^{(a)}]$  which only depends on  $\phi_1, \phi_2, \dots, \phi_N$ . In this way the functional  $F[\phi]$  is replaced by the function  $\tilde{F}(\phi_1, \phi_2, \dots, \phi_N) := F[\phi^{(a)}]$ . In a great desire to improve this approximation one take the limits where the lattice distance approaches zero and the hypercube  $\mathcal{C}$  approaches to the whole Minkowski spacetime  $\mathbf{M}$ . Let us symbolize both these limits by ' $N \rightarrow \infty$ '. Then the functional integral in (224) is introduced by the pseudo-definition

$$\int \mathcal{D}\phi F[\phi] := \lim_{N \rightarrow \infty} \underbrace{\int \dots \int}_N \tilde{F}(\phi_1, \phi_2, \dots, \phi_N) \prod_{i=1}^N d\phi_i \quad (226)$$

Although this is a lot of wishful mathematics, Feynman's functional integrals are a powerful tool to obtain quantitative results (sometimes agreeing excellently with experiment) and conceptional insights in quantum field theory.

Now the miracle of Feynman quantization. Recall that our classical system is characterized by its action functional  $S = S[\phi]$  defined by

$$S[\phi] := \int \mathcal{L}(\phi, \partial_\mu \phi) d^4x \quad (227)$$

This action gives rise to  $\tau$ -functions through

$$\langle 0|T(\underline{\phi}(x_1)\underline{\phi}(x_2)\dots\underline{\phi}(x_n))|0\rangle = \int \mathcal{D}\phi \phi(x_1)\phi(x_2)\dots\phi(x_n)e^{\frac{i}{\hbar}S[\phi]} \quad (228)$$

Observe that in the right-hand side, apart from  $\hbar := h/2\pi$  ( $h$  being Planck's constant) only entities from classical field theory occur. Nevertheless, this functional integral determines the  $\tau$ -functions of the corresponding quantum field theory, and therewith for instance all scattering processes for the system in question.

A related result is obtained when one considers a local observable  $\mathcal{O}(x) \equiv \mathcal{O}(\phi(x), \partial_\mu \phi(x))$  at the point  $x$ . In the corresponding quantum theory this is represented by the operator  $\underline{\mathcal{O}}(x) \equiv \mathcal{O}(\underline{\phi}(x), \partial_\mu \underline{\phi}(x))$  and one has

$$\langle 0|\underline{\mathcal{O}}(x)|0\rangle = \int \mathcal{D}\phi \mathcal{O}(x)e^{\frac{i}{\hbar}S[\phi]} \quad (229)$$

More generally, one has for a functional  $F = F[\phi]$  of the classical field  $\phi$

$$\langle 0|T(F[\underline{\phi}])|0\rangle = \int \mathcal{D}\phi F[\phi]e^{\frac{i}{\hbar}S[\phi]} \quad (230)$$

Observe that (228) is a particular case of (230) with  $F[\phi] = \phi(x_1)\phi(x_2)\dots\phi(x_n)$ .

The pendant of Feynman's functional integrals in the realm of one integration variable are integrals of the Fresnel type  $\int_{-\infty}^{\infty} f(x) \exp(ix^2) dx$ . By a rotation in the complex plane this can be transform in a Gaussian integral  $\int_{-\infty}^{\infty} \tilde{f}(x) \exp(-x^2) dx$  for which eventual convergence is much more easily asserted.

In quantum field theory one proceeds in a similar way. One performs the following substitution of the time coordinates, called a Wick rotation,

$$t \rightarrow \tau := -it \quad (231)$$

which entails

$$\phi(ct, \mathbf{x}) \rightarrow \phi_E(c\tau, \mathbf{x}) \quad (232)$$

$$\mathcal{L} \rightarrow \mathcal{L}_E := - \left[ \frac{1}{2}(\partial_\tau \phi_E)^2 + \frac{1}{2}\nabla \phi_E \cdot \nabla \phi_E + \frac{1}{2}m^2 \phi_E^2 + \mathcal{V}(\phi_E) \right] \quad (233)$$

and

$$\int \mathcal{D}\phi F[\phi]e^{\frac{i}{\hbar}S[\phi]} \rightarrow \int \mathcal{D}\phi F[\phi_E]e^{-\frac{1}{\hbar}S_E[\phi_E]} \quad (234)$$

where

$$S_E[\phi_E] := \int \left[ \frac{1}{2}(\partial_\tau \phi_E)^2 + \frac{1}{2}\nabla \phi_E \cdot \nabla \phi_E + \frac{1}{2}m^2 \phi_E^2 + \mathcal{V}(\phi_E) \right] d\tau d\mathbf{x} \quad (235)$$

Under a Wick rotation the Minkowski metric changes (up to a sign) into the euclidean metric:  $(ct)^2 - \mathbf{x}^2 \rightarrow -[(c\tau)^2 + \mathbf{x}^2]$ . The quantum field theory described by (234) is called a euclidean quantum field theory. Euclidean quantum field theories play an important role in the quest to put field theory on a mathematically sound foundation (constructive quantum field theory).

**U<sub>A</sub>(1) symmetry** We next consider some (absence of) symmetry in classical and quantum electrodynamics. The lagrangian of classical electrodynamics reads

$$\mathcal{L}(\psi, \bar{\psi}, A_\mu) = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{e^2}\text{Tr}(F_{\mu\nu}F^{\mu\nu}) \quad (236)$$

where  $\psi$  is the 4-component Dirac spinor field, describing after quantization e.g. electrons and positrons, and  $F_{\mu\nu}$  is the electromagnetic field strength tensor, describing after quantization photons. The electromagnetic field strength tensor  $F_{\mu\nu}$  is obtained from the electromagnetic potentials  $A_\mu$  via

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \quad (237)$$

The  $\gamma^\mu$ 's are the four Dirac matrices, defined as unitary  $4 \times 4$  matrices satisfying

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \quad (238)$$

Considering  $\psi$  as a 4-component column vector and  $\dagger$  denoting hermitian conjugation one defines

$$\bar{\psi} := \psi^\dagger \gamma^0 \quad (239)$$

Finally  $D_\mu$  is the gauge covariant derivative:

$$D_\mu := \partial_\mu - iA_\mu \quad (240)$$

It is well known, and easily verified, that the transformations

$$\psi \rightarrow \hat{\psi} := e^{i\varepsilon} \psi \quad (\varepsilon \in \mathbf{R}) \quad (241)$$

entailing

$$\bar{\psi} \rightarrow \hat{\bar{\psi}} = \bar{\psi} e^{-i\varepsilon} \quad (242)$$

are symmetry transformations. This gives rise to the Noether current

$$j^\mu := \bar{\psi} \gamma^\mu \psi \quad (243)$$

with the continuity equation (local conservation law)

$$\partial_\mu j^\mu = 0 \quad (244)$$

This leads to the conservation of electric charge [see (206)].

The symmetry transformations (241) form the group U(1) of unitary  $1 \times 1$  matrices (phase factors). The lagrangian (236) is even invariant under a larger group. Namely a group where the phase  $\varepsilon$  depends on the spacetime coordinates:  $\varepsilon = \varepsilon(x^\mu)$ . Indeed, one has

$$\mathcal{L}(\psi', \bar{\psi}', A'_\mu) = \mathcal{L}(\psi, \bar{\psi}, A_\mu) \quad (245)$$

under the transformations

$$\psi \mapsto \psi' := e^{i\varepsilon(x)}\psi \quad (246)$$

$$\bar{\psi} \mapsto \bar{\psi}' = e^{-i\varepsilon(x)}\bar{\psi} \quad (247)$$

$$A_\mu \mapsto A'_\mu := A_\mu + \partial_\mu \varepsilon \quad (248)$$

The transformation (246)—(248) is called a gauge transformation.

A first hint that one also has charge conservation in the quantized theory can be made plausible as follows. We consider a modification of the transformation (246)—(248) where  $A_\mu$  is left unchanged:

$$\psi \mapsto \tilde{\psi} := e^{i\varepsilon(x)}\psi \quad (249)$$

$$\bar{\psi} \mapsto \tilde{\bar{\psi}} = e^{-i\varepsilon(x)}\bar{\psi} \quad (250)$$

$$A_\mu \mapsto \tilde{A}_\mu := A_\mu \quad (251)$$

Due to the difference of (248) and (251) the transformations (249)—(251) do not leave the lagrangian invariant. One has

$$\mathcal{L}(\psi, \bar{\psi}, A_\mu) \rightarrow \mathcal{L}(\tilde{\psi}, \tilde{\bar{\psi}}, A_\mu) = \mathcal{L}(\psi, \bar{\psi}, A_\mu) - \bar{\psi}\gamma^\mu\psi\partial_\mu\varepsilon \quad (252)$$

Changing the names of the integration variables  $\psi$  and  $\bar{\psi}$  gives

$$\begin{aligned} \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A_\mu e^{\frac{i}{\hbar}S[\psi, \bar{\psi}, A_\mu]} &= \int \mathcal{D}\tilde{\psi}\mathcal{D}\tilde{\bar{\psi}}\mathcal{D}A_\mu e^{\frac{i}{\hbar}S[\tilde{\psi}, \tilde{\bar{\psi}}, A_\mu]} \\ &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A_\mu e^{\frac{i}{\hbar}\{S[\psi, \bar{\psi}, A_\mu] - \int \bar{\psi}\gamma^\mu\psi\partial_\mu\varepsilon d^4x\}} \end{aligned} \quad (253)$$

Hence

$$\begin{aligned} 0 &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A_\mu \int d^4x \bar{\psi}(x)\gamma^\mu\psi(x)\partial_\mu\varepsilon(x) e^{\frac{i}{\hbar}S[\psi, \bar{\psi}, A_\mu]} \\ &= \int d^4x \varepsilon(x) \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A_\mu \partial_\mu(\bar{\psi}(x)\gamma^\mu\psi(x)) e^{\frac{i}{\hbar}S[\psi, \bar{\psi}, A_\mu]} \end{aligned} \quad (254)$$

and this gives

$$\begin{aligned} 0 &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A_\mu \partial_\mu(\bar{\psi}(x)\gamma^\mu\psi(x)) e^{\frac{i}{\hbar}S} \\ &= \partial_\mu \langle 0 | \bar{\psi}(x)\gamma^\mu\psi(x) | 0 \rangle \end{aligned} \quad (255)$$

where the fields in the right-hand side are operator fields. Below we also will not explicitly indicate whether we have classical fields or operator fields, since this will be clear from the context. Similarly one can show that

$$\langle 0|T\left(\frac{\partial\bar{\psi}(x)\gamma^\mu\psi(x)}{\partial x^\mu}\dots\bar{\psi}_{a_i}(x_i)\dots\psi_{b_j}(y_j)\dots\psi_{b_m}(y_m)\dots A_{\mu_p}(z_p)\dots\right)|0\rangle = 0 \quad (256)$$

By means of the reduction formulas of Lehmann, Symanzik and Zimmermann one can conclude from this the operator relation

$$\frac{\partial}{\partial x^\mu}\bar{\psi}(x)\gamma^\mu\psi(x) = 0 \quad (257)$$

representing charge conservation in the quantized theory. It is to be noticed that the above derivation hinges on the invariance of the functional integration elements under the transformations (249)—(251). Formally one expects

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = e^{i\epsilon(x)}\mathcal{D}e^{-i\epsilon(x)}\psi\mathcal{D}\bar{\psi} = \mathcal{D}\psi\mathcal{D}\bar{\psi} \quad (258)$$

and this can be made plausible even after a more careful consideration of these functional integrations. A different state of affairs is encountered for the following transformations.

We now consider the lagrangian (236) with  $m = 0$ , i.e. massless quantum electrodynamics. For this we introduce the fifth gamma matrix

$$\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3 \quad (259)$$

This is a hermitian matrix since  $\gamma^0$  is hermitian and  $\gamma^1$ ,  $\gamma^2$  and  $\gamma^3$  are anti-hermitian. The transformations

$$\psi \rightarrow \hat{\psi} := e^{i\epsilon\gamma_5}\psi \quad (\epsilon \in \mathbf{R}) \quad (260)$$

which entails

$$\bar{\psi} \rightarrow \hat{\bar{\psi}} = \bar{\psi}e^{-i\epsilon\gamma_5} \quad (261)$$

are symmetry transformations in massless quantum electrodynamics. This gives rise to the conserved Noether current

$$j_A^\mu := \bar{\psi}\gamma_5\gamma^\mu\psi \quad (262)$$

with the continuity equation (local conservation law)

$$\partial_\mu j_A^\mu = 0 \quad (263)$$

This leads to the conservation of the so-called axial charge defined by

$$Q_A := \int \bar{\psi}\gamma_5\gamma^0\psi dx \quad (264)$$

and its conservation law reads

$$\frac{d}{dt}Q_A = 0 \quad (265)$$



The symmetry transformations (260) form a group of symmetry transformations denoted by  $U_A(1)$ . In order to investigate whether the axial charge is conserved we consider the transformations [compare (249) and (250)]

$$\psi(x) \mapsto \tilde{\psi}(x) := e^{i\epsilon(x)\gamma_5}\psi(x) \quad (266)$$

$$\bar{\psi}(x) \mapsto \bar{\tilde{\psi}}(x) = e^{-i\epsilon(x)}\bar{\psi}(x) \quad (267)$$

$$A_\mu(x) \mapsto \tilde{A}_\mu(x) := A_\mu(x) \quad (268)$$

For these transformations one has

$$\mathcal{L}(\psi, \bar{\psi}, A_\mu) \rightarrow \mathcal{L}(\tilde{\psi}, \bar{\tilde{\psi}}, A_\mu) = \mathcal{L}(\psi, \bar{\psi}, A_\mu) - \bar{\psi}\gamma^\mu\gamma_5\psi\partial_\mu\epsilon \quad (269)$$

Changing the names of the integration variables  $\psi$  and  $\bar{\psi}$  gives

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A_\mu e^{\frac{i}{\hbar}S[\psi, \bar{\psi}, A_\mu]} = \int \mathcal{D}\tilde{\psi}\mathcal{D}\bar{\tilde{\psi}}\mathcal{D}A_\mu e^{\frac{i}{\hbar}S[\tilde{\psi}, \bar{\tilde{\psi}}, A_\mu]} \quad (270)$$

However, a careful treatment of the functional integration element gives in this case

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} \rightarrow \mathcal{D}\tilde{\psi}\mathcal{D}\bar{\tilde{\psi}} = \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp -\frac{i}{16\pi^2} \text{Tr} \int \epsilon(x)\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}F_{\rho\sigma} \quad (271)$$

where  $\epsilon^{\mu\nu\rho\sigma}$  is the completely anti-symmetric Levi-Civita symbol  $\epsilon^{0123} = -1$ . The extra term in the right-hand side is the analogue of a Jacobian for functional intrals. All this leads to [compare (257)]

$$\frac{\partial}{\partial x^\mu} \overline{\psi(x)\gamma^\mu\gamma_5\psi(x)} = \frac{i}{16\pi^2} \text{Tr}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}F_{\rho\sigma} \quad (272)$$

Although in classical massless electrodynamics one has  $U_A(1)$  symmetry and conservation of the axial charge these features do not appear in the quantized theory due to the fact that the right-hand side of (272) is not zero. This is called an anomaly. Anomalies in quantum field theory have a long history. Their treatment in the setting of functional integrals is due to Fujikawa.

The purpose of the forgoing discussion of an anomaly for a simple model was the fact that they lead us to consider a bosonic string in 26-dimensional spacetime only. The lagrangian of the Polyakov string reads

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}g^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x^\nu\eta_{\mu\nu} \quad (273)$$

where  $x^\mu = x^\mu(\sigma, \tau)$ ,  $g_{\alpha\beta} = g_{\alpha\beta}(\sigma, \tau)$ ,  $g := \det(g_{\alpha\beta})$ ,  $\eta_{\mu\nu} := \text{diag}(1, -1, \dots, -1)$ ,  $(\xi^1, \xi^2) \equiv (\sigma, \tau)$  and  $\partial_\alpha \equiv \frac{\partial}{\partial \xi^\alpha}$ . This Lagrangian has a Weyl symmetry, i.e. it is invariant under the transformation

$$g^{\alpha\beta}(\sigma, \tau) \mapsto e^{\Lambda(\sigma, \tau)}g^{\alpha\beta}(\sigma, \tau) \quad (274)$$

$$x^\mu(\sigma, \tau) \mapsto x^\mu(\sigma, \tau) \quad (275)$$

It can be shown (see [5]) that the partition function  $Z$  of the Polyakov string has in the conformally Euclidean gauge  $g_{\alpha\beta}(\sigma, \tau) = \rho(\sigma, \tau)\delta_{\alpha\beta}$  the form

$$Z = \int \mathcal{D}\sqrt{\rho}\mathcal{D}\tilde{x}^\mu\mathcal{D}\xi\mathcal{D}\tilde{\eta} \exp \left\{ \int \left[ -\frac{1}{2}\partial_\alpha\tilde{x} \cdot \partial_\beta\tilde{x}\delta^{\alpha\beta} + \xi \partial\tilde{\eta} \right] d\sigma\tau + \Gamma_{\text{WZ}}(\phi) \right\} \quad (276)$$

where

$$\Gamma_{\text{WZ}}(\phi) = -\frac{26-n}{24\pi} \int \left[ \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\mu^2(e^{2\phi} - 1) \right] \quad (277)$$

is the Liouville action and  $\tilde{x} := \sqrt{\rho}x$ ,  $\tilde{\eta} := \rho\eta$  and  $n$  is the dimension of the Minkowski spacetime. Due to the Wess–Zumino term  $\Gamma_{\text{WZ}}$  we have, in general, lost the conformal invariance by quantizing the Polyakov string. However, if dimension of the Minkowski spacetime  $n = 26$  the anomaly is absent and the quantized theory is conformally invariant as well. If one wants the survival of the idea of a string at all in the act of quantization it seems that one has to restrict to 26–dimensional spacetime. Since the immediately observable spacetime is four–dimensional 22 of the 26 dimensions have to be made unobservable. This can be done in a Kaluza–Klein type of way, by curling these extra dimensions up so that their spacial extensions are too small for observation at present. Compared with the original Kaluza–Klein theory a string brings with it a new feature, a winding number. The curling up of the extra dimensions is called compactification. In the next section there is made a modest start by first curling up only one of these extra dimensions. Thereafter we turn to the question of curling up more dimensions by letting them form a torus.

#### 4. TOROIDAL COMPACTIFICATION

We now turn to the compactification of 26–dimensional Minkowski spacetime  $\mathbf{M}^{26}$ . That is, we will consider string theory in a spacetime  $\mathbf{M}^k \times \mathcal{C}^{26-k}$  with  $\mathbf{M}^k$  being  $k$ –dimensional Minkowski spacetime and  $\mathcal{C}^{26-k}$  a  $(26 - k)$ –dimensional compact riemannian manifold. Actually we will take here  $\mathcal{C}^{26-k} = T^{26-k}$ , a  $(26 - k)$ –dimensional torus. In order to simplify things we start with the case  $k = 25$ . This is the case where one spatial direction of  $\mathbf{M}^{26}$ , e.g.  $x^1$ , is turned into a circle  $S^1$  (with radius  $R$ ) and the spacetime manifold  $M$  becomes a 26–dimensional cylinder

$$M = \mathbf{M}^{26}/2\pi R\mathbf{Z} = \mathbf{M}^{25} \times S^1 \quad (278)$$

More explicitly, spacetime  $M$  arises by identifying in the 26–dimensional Minkowski spacetime the points  $(x^0, x^1, x^2, \dots, x^{25})$  and  $(x^0, x^1 + 2\pi R s, x^2, \dots, x^{25})$  ( $s \in \mathbf{Z}$ ). Instead of  $R$  we will frequently use  $a$  defined by  $R =: \ell a$ . A closed string can wind several times around the cylinder (see figure 1). Hence the condition that the string is closed reads now

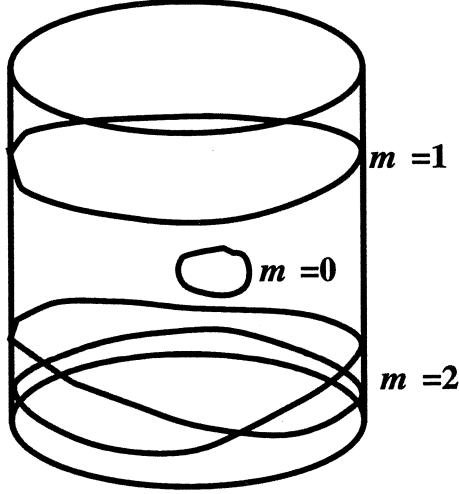


FIGURE 1. Winding numbers of a string on a cylinder.

$$x^1(\sigma + 2\pi, \tau) = x^1(\sigma, \tau) + 2\pi m R \quad (m \in \mathbf{Z}) \quad (279)$$

where  $m$  is called the winding number. The general solution of the equation of motion of  $x^1$  is given by

$$x^1(\sigma, \tau) = q^1 + \frac{1}{2}(p_L^1 + p_R^1)\tau + \frac{1}{2}(p_L^1 - p_R^1)\sigma + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \left[ \alpha_n^{L1} e^{in(\tau+\sigma)} + \alpha_n^{R1} e^{in(\tau-\sigma)} \right] \quad (280)$$

This gives with the condition (279)

$$p_L^1 - p_R^1 = 2ma \quad (281)$$

Hence in the Schrödinger representation these operators are given by

$$p_L^1 = \frac{1}{2i} \frac{\partial}{\partial q^1} + ma \quad p_R^1 = \frac{1}{2i} \frac{\partial}{\partial q^1} - ma \quad (282)$$

The operator  $p^1 = p_L^1 + p_R^1 = \frac{1}{i} \frac{\partial}{\partial q^1}$  has an eigenfunction  $\exp(ik^1 q^1)$  with eigenvalue  $k^1 \in \mathbf{R}$ . Wave functions have to be single-valued, i.e.  $\exp(ik^1 q^1)$  has to be invariant under  $x^1 \rightarrow x^1 + 2\pi R$  or  $q^1 \rightarrow q^1 + 2\pi R/\ell = q^1 + 2\pi a$ . This condition gives

$$k^1 a = n \quad (n \in \mathbf{Z}) \quad (283)$$

The eigenvalue equations

$$p_L^1 \exp(ik^1 q^1) = k_L^1 \exp(ik^1 q^1), \quad (284)$$

$$p_R^1 \exp(ik^1 q^1) = k_R^1 \exp(ik^1 q^1) \quad (k_L^1, k_R^1 \in \mathbf{R})$$

imply [see (282)]

$$k_L^1 = \frac{n}{2a} + ma \quad k_R^1 = \frac{n}{2a} - ma \quad (285)$$

The corresponding eigenvalues of  $P_L^1$  and  $P_R^1$  are respectively  $m \frac{RT\pi}{c} + n \frac{1}{2R}$  and  $-m \frac{RT\pi}{c} + n \frac{1}{2R}$  since

$$P_L^1 = \left( \frac{\pi T}{c} \right)^{\frac{1}{2}} p_L^1, \quad P_R^1 = \left( \frac{\pi T}{c} \right)^{\frac{1}{2}} p_R^1 \quad (286)$$

The conditions (279) and (283) have a quite different type of origin: (283) has a quantum mechanical background, whereas (279), depending on the winding number, has a classical origin. The tuning of these two effects will be seen to give rise to the Frenkel–Kac–Segal mechanism.

The mass-squared has to do with the motion in the 25-dimensional Minkowski spacetime:

$$M^2 := \sum_{\mu \neq 1} P_\mu P^\mu = \frac{4\pi T}{c} \sum_{\mu \neq 1} p_\mu p^\mu \quad (287)$$

By restricting to the subspace of the physical states one has

$$1 = \frac{1}{2} \left[ (p_L^1)^2 + \sum_{\mu \neq 1} p_{L\mu} p_L^\mu \right] = \frac{1}{2} \left[ (p_R^1)^2 + \sum_{\mu \neq 1} p_{R\mu} p_R^\mu \right] \quad (288)$$

This gives, using (287),

$$M^2 = \frac{4\pi T}{c} [N_L + N_R + \frac{1}{2}(p_R^1)^2 + \frac{1}{2}(p_L^1)^2 - 2] \quad (289)$$

The eigenvalues  $m^2$  of  $M^2$  are given by [see (285)]

$$m^2 = \frac{4\pi T}{c} [N'_L + N'_R + \left( \frac{n}{2a} \right)^2 + (ma)^2 - 2] \quad (290)$$

where  $N'_L$  and  $N'_R$  are eigenvalues of  $N_L$  and  $N_R$ . Let  $|k, k_L, k_R\rangle$  be a state without excitations, i.e.

$$N_L |k, k_L, k_R\rangle = 0, \quad N_R |k, k_L, k_R\rangle = 0 \quad (291)$$

$$P_L^\mu |k, k_L, k_R\rangle = k_L^\mu |k, k_L, k_R\rangle, \quad P_R^\mu |k, k_L, k_R\rangle = k_R^\mu |k, k_L, k_R\rangle \quad (\mu \neq 1) \quad (292)$$

$$P_L^1 |k, k_L, k_R\rangle = k_L^1 |k, k_L, k_R\rangle, \quad P_R^1 |k, k_L, k_R\rangle = k_R^1 |k, k_L, k_R\rangle \quad (293)$$

From (290) we see that there massless states for  $N'_L = N'_R = 1, n = m = 0$ . In general, there are actually two independent state vectors for these states:

$$\alpha_{-1}^{L-1} \alpha_{-1}^{R-\mu} |k, k_L, k_R\rangle \quad (\mu \neq 1) \quad (294)$$

and

$$\alpha_{-1}^{R-1} \alpha_{-1}^{L-\mu} |k, k_L, k_R\rangle \quad (\mu \neq 1) \quad (295)$$

This are massless bosons with helicity one. One contends that they are the particle states of gauge fields.

In passing we note that the two-dimensional vectors  $(p_L^1, p_R^1) = (\frac{n}{2a} + ma, \frac{n}{2a} - ma)$  form a lattice

$$\Lambda := \{ne_1 + me_2 \mid e_1 := (\frac{1}{2a}, \frac{1}{2a}), e_2 := (a, -a), n, m \in \mathbf{Z}\} \quad (296)$$

A lattice is called even if  $p^2$  is an even integer for each lattice vector  $p$ . Since

$$(p_L^1, p_R^1)^2 := (p_L^1)^2 - (p_R^1)^2 = 2mn \in 2\mathbf{Z} \quad (297)$$

the lattice  $\Lambda$  is even. The dual basis  $\{e_1^*, e_2^*\}$  of the basis  $\{e_1, e_2\}$  is defined by

$$e_i \cdot e_j^* = \delta_{ij} \quad (298)$$

where the dot  $\cdot$  indicates the two-dimensional Minkowski scalar product [compare (297)]. This gives

$$e_1^* = (a, -a) = e_2, \quad e_2^* := (\frac{1}{2a}, \frac{1}{2a}) = e_1 \quad (299)$$

The dual lattice of  $\Lambda$  is defined by

$$\Lambda^* := \{ne_1^* + me_2^* \mid n, m \in \mathbf{Z}\} \quad (300)$$

A lattice is called selfdual if  $\Lambda^* = \Lambda$ . From (299) one sees that the lattice  $\Lambda$  is selfdual.

The term  $(\frac{n}{2a})^2 + (ma)^2$  in (290) is due to the  $S^1$  compactification. Recall that its first term had a quantum mechanical origin (single-valuedness of a wave function) and its second term a classical origin (winding number). The map  $a \mapsto \frac{1}{2a}$  entails  $\frac{1}{2a} \mapsto a$  and  $(\frac{n}{2a})^2 + (ma)^2 \mapsto (na)^2 + (\frac{m}{2a})^2$ . Hence this boils down to an interchange of  $m$  and  $n$ . For  $a = 1/2a$ , i.e.

$$a = \frac{1}{\sqrt{2}}, \text{ or } R = \sqrt{\frac{c}{2\pi T}} \quad (301)$$

this map has a fixed point and the mass formula (290) becomes in this case

$$m^2 = \frac{4\pi T}{c} [N'_L + N'_R + \frac{1}{2}(n^2 + m^2) - 2] \quad (302)$$

where  $n$  and  $m$  now play an equivalent role. The choice (301) of a specific radius of the compactifying circle entails new massless states [see (302)] with  $N'_L = 0, N'_R = 1, n^2 + m^2 = 2$  and  $N'_L = 1, N'_R = 0, n^2 + m^2 = 2$ . From (285) we now get  $(k_L^1, k_R^1) = \frac{1}{\sqrt{2}}(n + m, n - m)$ . Since  $n^2 + m^2 = 2$  we have  $n, m = \pm 1$  and thus we arrive at  $(k_L^1, k_R^1) = (\pm\sqrt{2}, 0)$  or  $(0, \pm\sqrt{2})$ . Thus the four state vectors of the extra massless states are

$$\alpha_{-1}^{R\ \mu} |k, \pm\sqrt{2}, 0\rangle \quad (\mu \neq 1) \quad (303)$$

and

$$\alpha_{-1}^{L\ \mu} |k, 0, \pm\sqrt{2}\rangle \quad (\mu \neq 1) \quad (304)$$

So we end up in this special case with  $2+4 = 6$  massless helicity-one states. We will see that these are the particles of the six gauge fields of the 6-dimensional Lie group  $SU(2) \times SU(2)$ .

### 5. FRENKEL-KAC-SEGAL MECHANISM

We now turn to the compactification of several dimensions. In particular we will consider instead of  $M^{26}$  a spacetime

$$M = M^{26}/\Lambda \quad (305)$$

where  $\Lambda$  is a lattice spanning a  $d$ -dimensional euclidean (linear) submanifold of  $M^{26}$ . The replacement  $M^{26} \rightarrow M$  is called toroidal compactification. Hence

$$M = M^{26-d} \times T^d \quad (306)$$

where  $T^d$  is a  $d$ -dimensional torus. Thus we identify the points  $(x^0, \dots, x^{25-d}, x^{26-d}, \dots, x^{25})$  and  $(x^0, \dots, x^{25-d}, x^{26-d} + \lambda^{26-d}, \dots, x^{25} + \lambda^{25})$  if  $(\lambda^{26-d}, \dots, \lambda^{25}) \in \Lambda$ . For the sake of convenience we consider only the components of  $x^\mu = x^\mu(\sigma, \tau)$  for  $\mu = 26-d, \dots, 25$ , i.e. the motion of the string on the torus. Likewise, we restrict ourselves to the consideration of either left-movers or right-movers, however, we will suppress the subscripts  $L$  and  $R$ . The resulting  $d$ -component operator field will be denoted by [compare (170) and (171)]

$$x^\mu = x^\mu(z) \quad (\mu = 26-d, \dots, 25, \quad z := e^{i(\tau \pm \sigma)}) \quad (307)$$

Defining

$$x^\mu_> = i \sum_{n>0} \frac{\alpha_n^\mu}{n} z^{-n} \quad (308)$$

$$x^\mu_< = i \sum_{n<0} \frac{\alpha_n^\mu}{n} z^{-n} \quad (309)$$

$$x^\mu_0 = q^\mu - ip^\mu \log z \quad (310)$$

one has

$$x^\mu = x^\mu_< + x^\mu_0 + x^\mu_> \quad (311)$$

The so-called vertex operator is defined by

$$U(\gamma, z) := \exp(i\gamma_\mu x^\mu_<) \exp(i\gamma_\mu x^\mu_0) \exp(i\gamma_\mu x^\mu_>) \quad (312)$$

Below we need to discuss whether  $U(\gamma, z)$  is a single-valued function of  $z$ . The potentially dangerous term appears to be the second factor in the right-hand side of (312) [see (310)]. By means of the operator identity

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]} \quad (313)$$

which holds if  $[A, B] = \lambda \mathbf{1}$  ( $\lambda \in \mathbf{C}$ ) we can rewrite this factor as

$$e^{i\gamma \cdot x_0(z)} = e^{i\gamma \cdot q} e^{\gamma \cdot p \log z} e^{\frac{1}{2}q^2} \quad (314)$$

where  $a_\mu b^\mu \equiv a \cdot b$  and  $a^2 \equiv a \cdot a$ .

The generators of the Virasoro algebra  $L_n$  have the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{d}{12}m(m^2-1)\delta_{m,-n} \quad (m, n \in \mathbf{Z}) \quad (315)$$

where  $d = \dim \Lambda$ . The  $L_n$ 's generate conformal transformations. The transformation of the vertex operator under conformal transformations is given by

$$[L_n, U(\gamma, z)] = \left( z^{n+1} \frac{d}{dz} - n z^n \frac{\gamma^2}{2} \right) U(\gamma, z) \quad (316)$$

For the tachyon with  $\gamma^2 = -2$  this gives

$$[L_n, U(\gamma, z)] = z^1 \frac{d}{dz} (z^n U(\gamma, z)) \quad (317)$$

We now introduce

$$A_\gamma := \frac{1}{2\pi i} \oint U(\gamma, z) \frac{dz}{z} \quad (\gamma^2 = -2) \quad (318)$$

where the contour is a closed curve winding once around the origin  $z = 0$  of the complex plane, e.g. a circle. From (317) one obtains then

$$[L_n, A_\gamma] = \frac{1}{2\pi i} \oint \frac{d}{dz} (z^n U(\gamma, z)) dz \quad (319)$$

Hence  $[L_n, A_\gamma] = 0$  if  $U(\gamma, z)$  is single-valued. In order to investigate this question we look at the action of  $e^{i\gamma \cdot x_0(z)}$  on

$$|\lambda\rangle := e^{iq \cdot \lambda} |0\rangle \quad (320)$$

We find, using (313),

$$\begin{aligned} e^{i\gamma \cdot x_0(z)} |\lambda\rangle &= e^{i\gamma \cdot (q - ip \log z)} e^{-iq \cdot \gamma} |\lambda + \gamma\rangle = e^{\gamma \cdot p \log z + \frac{1}{2}[\gamma \cdot p \log z, -iq \cdot \gamma]} |\lambda + \gamma\rangle \\ &= e^{\gamma \cdot (\lambda + \gamma) \log z + -\log z} |\lambda + \gamma\rangle \end{aligned} \quad (321)$$

or

$$e^{i\gamma \cdot x_0(z)} |\lambda\rangle = z^{1+\gamma \cdot \lambda} |\lambda + \gamma\rangle \quad (322)$$

In order that  $e^{i\gamma \cdot x_0(z)}$  is single-valued we require

$$\gamma \cdot \lambda \in \mathbf{Z} \quad (323)$$

A lattice  $\Lambda$  in a vector space with a bilinear form, denoted by  $\cdot$ , is called integral if for all  $\gamma, \lambda \in \Lambda$  one has  $\gamma \cdot \lambda \in \mathbf{Z}$ . The requirement that the lattice which is used for the toroidal compactification is even leads, via the single-valuedness of  $U(\gamma, z)$ , to

$$[L_n, A_\gamma] = 0 \quad (\gamma \in \Lambda^{(2)}) \quad (324)$$

where  $\Lambda^2 := \{\gamma \in \Lambda \mid \gamma^2 = -2\}$  with  $\Lambda$  an integral lattice.

An important consequence of (324) is that  $A_\gamma$  maps physical states to physical states. Indeed, let  $|\psi\rangle$  be a physical state, i.e.  $L_n|\psi\rangle = 0$  ( $n = 1, 2, \dots$ ) and  $L_0|\psi\rangle = |\psi\rangle$  then  $L_n(A_\gamma|\psi\rangle) = 0$  ( $n = 1, 2, \dots$ ) and  $L_0(A_\gamma|\psi\rangle) = (A_\gamma|\psi\rangle)$ . Hence  $A_\gamma|\psi\rangle$  is a physical state. One can show that

$$A_\beta A_\gamma - (-1)^{\beta \cdot \gamma} A_\gamma A_\beta = \begin{cases} A_{\beta+\gamma} & \text{if } \beta \cdot \gamma = -1 \\ \beta \cdot p & \text{if } \beta \cdot \gamma = -2 \\ 0 & \text{otherwise} \end{cases} \quad (325)$$

$$[p^\mu, A_\gamma] = \gamma^\mu A_\mu \quad (326)$$

$$[p^\mu, p^\nu] = 0 \quad (327)$$

The remaining obstacle on our way towards a Lie algebra which transforms physical states into physical states is the factor  $(-1)^{\beta \cdot \gamma}$  in (325). So one introduces operators

$$E^\gamma := A_\gamma c_\gamma \quad (328)$$

and one adjusts the operators  $c_\gamma$  in such a way that the undesirable factors  $(-1)^{\beta \cdot \gamma}$  in (325) disappear. That this is indeed possible is now explained in a number of steps. First, one writes

$$c_\gamma := e^{i\gamma \cdot q} \hat{c}_\gamma \quad (329)$$

and defines  $\hat{c}_\gamma$  by

$$\hat{c}_\gamma|\lambda\rangle := \varepsilon(\gamma, \lambda)|\gamma + \lambda\rangle \quad (330)$$

Second,  $\varepsilon(\gamma, \lambda)$  is taken to be a function on the lattice  $\Lambda$  with values  $\pm 1$  such that

$$\varepsilon(\gamma, \lambda)\varepsilon(\gamma + \lambda, \delta) = \varepsilon(\gamma, \lambda + \delta)\varepsilon(\gamma, \delta) \quad (331)$$

called a cocycle condition, and

$$\varepsilon(\gamma, \lambda) = (-1)^{\beta \cdot \gamma} \varepsilon(\lambda, \gamma) \quad (332)$$

All this implies



$$\hat{c}_\beta \hat{c}_\gamma |\lambda\rangle = \varepsilon(\beta, \gamma) \hat{c}_{\beta+\gamma} |\lambda\rangle |\gamma + \lambda\rangle \quad (333)$$

or

$$\hat{c}_\beta \hat{c}_\gamma = \varepsilon(\beta, \gamma) \hat{c}_{\beta+\gamma} \quad (334)$$

Using (332) this gives

$$\hat{c}_\beta \hat{c}_\gamma - (-1)^{\beta \cdot \gamma} \hat{c}_\gamma \hat{c}_\beta = 0 \quad (335)$$

A cocycle is not unique, since  $\eta_\gamma \eta_\lambda \varepsilon(\gamma, \lambda)$  is a cocycle if  $\varepsilon(\gamma, \lambda)$  is a cocycle. This freedom in the choice of a cocycle allows us to require additionally

$$\varepsilon(\gamma, -\gamma) = \varepsilon(\gamma, 0) = 1 \quad (336)$$

for all  $\gamma \in \Lambda$ .

Finally we find in this way a Lie algebra  $g_\Lambda$  spanned by  $\{E^\gamma, p^\mu \mid \gamma \in \Lambda^{(2)}, 1 \leq \mu \leq \dim \Lambda\}$  with commutation relations

$$[E^\beta, E^\gamma] = \begin{cases} \varepsilon(\beta, \gamma) E^{\beta+\gamma} & \text{if } \beta \cdot \gamma = -1 \\ \beta \cdot p & \text{if } \beta = -\gamma \\ 0 & \text{otherwise} \end{cases} \quad (337)$$

$$[p^\mu, E^\beta] = \beta^\mu E^\beta \quad (338)$$

$$[p^\mu, p^\nu] = 0 \quad (339)$$

Notice that  $\dim g_\Lambda = \#(\Lambda^{(2)}) + \dim \Lambda$  where  $\#(\Lambda^{(2)})$  is the number of elements of  $\Lambda^{(2)}$ , the set  $\{p^\mu \mid 1 \leq \mu \leq \dim \Lambda\}$  spans a Cartan subalgebra and for all simple roots one has  $\|\beta\| = \sqrt{2}$ , i.e.  $g_\Lambda$  is a simply-laced Lie algebra.

In some physical models the Virasoro generator  $L_0$  is the hamiltonian. Since the elements of  $g_\Lambda$  commute with  $L_0$  and transform physical states into physical states the Lie algebra  $g_\Lambda$  gives via exponentiation rise to a symmetry group.

This is the first part of the Frenkel-Kac-Segal construction. We now proceed with the higher moments

$$E_n^\beta := \frac{1}{2\pi i} \oint U(\beta, z) c_\beta z^n \frac{dz}{z} \quad (\beta \in \Lambda^{(2)}, n \in \mathbf{Z}) \quad (340)$$

$$\alpha_n^\mu := \frac{1}{2\pi i} \oint P^\mu(z) z^n \frac{dz}{z} \quad (1 \leq \mu \leq \dim \Lambda, n \in \mathbf{Z}) \quad (341)$$

where

$$P^\mu(z) := iz \frac{dx^\mu}{dz} = \sum_{n=-\infty}^{\infty} \alpha_n z^n \quad (342)$$

and the contours in (340) and (341) are e.g. circles around the origin  $z = 0$ . Observe that one has for  $n = 0$

$$E_0^\beta = E^\beta \quad \alpha_0^\mu = p^\mu \quad (343)$$

The Lie algebra spanned by  $\{E_n^\gamma, \alpha_n^\mu \mid \gamma \in \Lambda^{(2)}, 1 \leq \mu \leq \dim \Lambda\}$  is denoted by  $\hat{g}_\Lambda$ . Its commutation relations read

$$[E_m^\beta, E_n^\gamma] = \begin{cases} \varepsilon(\beta, \gamma) E_{m+n}^{\beta+\gamma} & \text{if } \beta \cdot \gamma = -1 \\ \beta \cdot \alpha_{m+n} + m \delta_{m, -n} & \text{if } \beta = -\gamma \\ 0 & \text{otherwise} \end{cases} \quad (344)$$

$$[\alpha_m^\mu, E_n^\beta] = \beta^\mu E_{m+n}^\beta \quad (345)$$

$$[\alpha_m^\mu, \alpha_n^\nu] = m \delta^{\mu\nu} \delta_{m, -n} \quad (346)$$

Hence  $\hat{g}_\Lambda$  is a Kac-Moody algebra.

The commutation relations of the generators of  $\hat{g}_\Lambda$  with the generators of the Virasoro algebra are given by

$$[L_m, E_n^\beta] = -n E_{m+n}^\beta \quad (347)$$

$$[L_m, \alpha_n^\mu] = -n \alpha_{m+n}^\mu \quad (348)$$

Denoting the generators of  $\hat{g}_\Lambda$  collectively by  $X_n$ , i.e.

$$X_n = E_n^\beta, \alpha_n^\mu \quad (349)$$

then for any eigenstate  $|h\rangle$  of  $L_0$  all non-zero vectors  $X_n|h\rangle$  are also eigenstates of  $L_0$ . More explicitly  $L_0|h\rangle = h|h\rangle$  and  $[L_0, X_n] = -n X_n$  imply  $-n(X_n|h\rangle) = L_0 X_n|h\rangle - X_n L_0|h\rangle = L_0 X_n|h\rangle - X_n h|h\rangle$  or

$$L_0(X_n|h\rangle) = (h - n)(X_n|h\rangle) \quad (350)$$

Since  $\hat{g}_\Lambda$  connects some of the energy eigenstates it is called a partially spectrum generating algebra.

One can argue that a lattice is suitable for toral compactification if it is even and self-dual. The dimension of even and self-dual euclidean lattices is  $\dim \Lambda = 8n$ . This is the number of dimensions which are compactified. Hence we compactify 8, 16, 24, ... dimensions. Since we start with 26-dimensional Minkowski spacetime we end up after toral compactification eventually with a spacetime with 18, 10 and 2 dimensions. The case of 10 dimensions is particularly interesting since a superstring requires a 10-dimensional Minkowski spacetime.

In this case we are interested in 16-dimensional even self-dual euclidean lattices. There are only two lattices of this type. They are called  $\Gamma^8 \oplus \Gamma^8$  and  $\Gamma^{16}$ . Here  $\Gamma^8$  is the root lattice of the Lie algebra  $E^8$  and  $\Gamma^{16}$  is the root lattice of the Lie algebra  $so(32)$  with one additional point or equivalently the weight lattice of the Lie algebra of the Lie group  $Spin(32)/\mathbf{Z}_2$ . Thus we finally arrive at the following result. Toroidal compactification of the bosonic string in 26-dimensional Minkowski spacetime by a suitable 10-dimensional lattice gives rise to either  $E_8 \times E_8$  or  $Spin(32)/\mathbf{Z}_2$  as symmetry group. The zero mass modes gives the gauge fields of these groups.

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# An Introduction to Hopf Algebras

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## 1. DEFINITION

The notion “algebra” can be described as follows.

A linear space  $A$  over a field  $K$  is an *algebra* over  $K$  or  $K$ -algebra if there is a *linear* mapping  $m: A \otimes_K A \rightarrow A$ . This mapping is called *multiplication*, for  $x$  and  $y \in A$  one mostly writes  $xy$  for  $m(x \otimes y)$ .

The algebra  $A$  is *associative* iff the following diagram commutes.

$$\begin{array}{ccc}
 A \otimes A \otimes A & \xrightarrow{m \otimes \text{Id}_A} & A \otimes A \\
 \text{Id}_A \otimes m \downarrow & & \downarrow m \\
 A \otimes A & \xrightarrow{m} & A
 \end{array}$$

$\text{Id}_A$  is the identity map on  $A$ .

$A$  is *commutative* iff the next diagram commutes.

$$\begin{array}{ccc}
 A \otimes A & \xrightarrow{\tau} & A \otimes A \\
 m \downarrow & & \downarrow m \\
 A & \xrightarrow{\text{Id}_A} & A
 \end{array}$$

$\tau$  is the so-called *twist* map;  
 $\tau$  is linear with  $\tau(x \otimes y) = y \otimes x$ .

$A$  has a unit iff the next two diagrams commutes.

$$\begin{array}{ccc}
 K \otimes A & \xrightarrow{i \otimes \text{Id}_A} & A \otimes A \\
 \downarrow j & & \downarrow m \\
 A & \xrightarrow{\text{Id}_A} & A
 \end{array}$$

$j$  is the linear map with  
 $j(1 \otimes x) = x$ ;  
 $i$  is the embedding of  $K$  into  
 $A$  ( $i(1) = 1$ , the units of  $K$   
 and  $A$  are identified).

$$\begin{array}{ccc}
 A \otimes K & \xrightarrow{\text{Id}_A \otimes i} & A \otimes A \\
 \downarrow j' & & \downarrow m \\
 A & \xrightarrow{\text{Id}_A} & A
 \end{array}$$

$j'$  is the linear map with  
 $j'(x \otimes 1) = x$ .

Let  $B$  another algebra and  $f: A \rightarrow B$  an application.  $f$  is a homomorphism (or  
 an algebra morphism) iff the following diagram commutes.

$$\begin{array}{ccc}
 A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\
 \downarrow m & & \downarrow m' \\
 A & \xrightarrow{f} & B
 \end{array}$$

$m'$  is the multiplication of  $B$ .

The notion “coalgebra” is the dual of the notion “algebra”. So, a linear space  
 $H$  over  $K$  is called a *coalgebra* if there is a linear mapping  $\Delta: H \rightarrow H \otimes H$ .  
 $\Delta$  is named *coproduct* or *diagonalisation*.  $H$  is said to be *coassociative* if the  
 following diagram commutes.

$$\begin{array}{ccc}
 H & \xrightarrow{\Delta} & H \otimes H \\
 \downarrow \Delta & & \downarrow \text{Id}_H \otimes \Delta \\
 H \otimes H & \xrightarrow{\Delta \otimes \text{Id}_H} & H \otimes H \otimes H
 \end{array}$$

$H$  is said to be *cocommutative* if the next diagram commutes.

$$\begin{array}{ccc}
 H & \xrightarrow{\text{Id}_H} & H \\
 \Delta \downarrow & & \downarrow \Delta \\
 H \otimes H & \xrightarrow{\tau} & H \otimes H
 \end{array}$$

$\tau$  is the already known twistmap.

$H$  has a so-called *counit*  $\epsilon$  if the next two diagrams commute.

$$\begin{array}{ccc}
 H & \xrightarrow{\text{Id}_H} & H \\
 \Delta \downarrow & & \downarrow h \\
 H \otimes H & \xrightarrow{\epsilon \otimes \text{Id}_H} & K \otimes H
 \end{array}$$

$h$  is the linear map with  $h(x) = 1 \otimes x$ .

$$\begin{array}{ccc}
 H & \xrightarrow{\text{Id}_H} & H \\
 \Delta \downarrow & & \downarrow h' \\
 H \otimes H & \xrightarrow{\text{Id}_H \otimes \epsilon} & H \otimes K
 \end{array}$$

$h'$  is the linear map with  $h'(x) = x \otimes 1$ .

Let  $H'$  be another coalgebra and  $f : H \rightarrow H'$  an application.  $f$  is said to be a *coalgebra morphism* if the following diagram commutes.

$$\begin{array}{ccc}
 H & \xrightarrow{f} & H' \\
 \Delta \downarrow & & \downarrow \Delta' \\
 H \otimes H & \xrightarrow{f \otimes f} & H' \otimes H'
 \end{array}$$

$\Delta'$  is the coproduct of  $H'$ .

A *bialgebra*  $H$  is an associative algebra as well as a coassociative coalgebra in such a way that both the coproduct  $\Delta$  and the counit  $\epsilon$  (always existing) are algebra morphisms. If  $H'$  is another bialgebra then  $f : H \rightarrow H'$  is a *bialgebra morphism* if  $f$  is an algebra- as well as a coalgebra morphism with  $\epsilon' \circ f = \epsilon$  where  $\epsilon'$  is the counit of  $H'$ .

REMARK. We just said that  $\Delta$  should be a homomorphism in order that  $H$  will be a bialgebra. This means that  $H \otimes H$  must be an algebra with a multiplication  $m'$  and that there must be a commutative diagram as below.

$$\begin{array}{ccc}
 H \otimes H & \xrightarrow{\Delta \otimes \Delta} & H \otimes H \otimes H \otimes H \\
 \downarrow m & & \downarrow m' \\
 H & \xrightarrow{\Delta} & H \otimes H
 \end{array}$$

Now, if  $A$  and  $B$  are associative algebras with unit 1 then  $A \otimes_K B$  is also an associative algebra with unit  $1 \otimes 1$ . The multiplication is given by  $(a \otimes b)(a' \otimes b') = (aa') \otimes (bb')$  so that  $m' = (m \otimes m) \circ (\text{Id}_H \otimes \tau \otimes \text{Id}_H)$ .

REMARK.  $H \otimes H$  is also a coalgebra with coproduct  $\Delta'$  equal to  $(\text{Id}_H \otimes \tau \otimes \text{Id}_H) \circ (\Delta \otimes \Delta)$ . With this definition  $m$  becomes a *coalgebramorphism*.

REMARK.  $K$  is a bialgebra with coproduct  $\Delta_K : K \rightarrow K \otimes_K K$  linear for which  $\Delta_K(1) = 1 \otimes 1$  (the inverse of the multiplication). With this in mind and with help of the definition one sees that a counit of a coalgebra  $C$  is a coalgebramorphism and even a bialgebramorphism if  $C$  is a bialgebra.

#### CONVOLUTION

Let  $A$  be an associative algebra with unit 1 and  $C$  a coassociative coalgebra with counit  $\epsilon$ . If  $u : C \rightarrow A$  and  $v : C \rightarrow A$  are linear then we define  $u \star v$  to be the linear map  $C \xrightarrow{\Delta} C \otimes C \xrightarrow{u \otimes v} A \otimes A \xrightarrow{m} A$ . In this way we get an associative multiplication, named "convolution", in  $\mathcal{L}(C; A)$ . There is a unit  $\mathbf{1}_{C,A} = i \circ \epsilon$  where  $i : K \rightarrow A$  is the linear map with  $i(1) = 1$ . This implies that counits (if existing) of coalgebras are unique (take  $A = K$ ).

#### ANTIPODE

Let now  $C = A = H$  be a bialgebra. The inverse  $s$  of  $\text{Id}_H$  (if existing) with respect to the convolution is called "antipode" or "inversion".

We shall state and prove some properties of  $s$ .

1.  $s(xy) = s(y)s(x)$  for  $x$  and  $y \in H$ . In other words

$$s \circ m = m \circ (s \otimes s) \circ \tau. \quad (1)$$

Proof. We shall prove the following formula.

$$(s \circ m) \star m = \mathbf{1}_{H \otimes H, H} = m \star (m \circ (s \otimes s) \circ \tau). \quad (2)$$



$$(s \circ m) \star m : H \otimes H \xrightarrow{\Delta_{H \otimes H}} H \otimes H \otimes H \otimes H \xrightarrow{(s \circ m) \otimes m} H \otimes H \xrightarrow{m} H. \quad (3)$$

Now  $\Delta_{H \otimes H} = (\text{Id} \otimes \tau \otimes \text{Id}) \circ (\Delta \otimes \Delta)$  and  $(s \circ m) \otimes m = (s \otimes \text{Id}_H) \circ (m \otimes m)$  moreover the multiplication  $m_{H \otimes H}$  of  $H \otimes H$  is given by  $(m \otimes m) \circ (\text{Id}_H \otimes \tau \otimes \text{Id}_H)$ . So (3) can be rewritten as

$$H \otimes H \xrightarrow{\Delta \otimes \Delta} H \otimes H \otimes H \otimes H \xrightarrow{m_{H \otimes H}} H \otimes H \xrightarrow{s \otimes \text{Id}} H \otimes H \xrightarrow{m} H. \quad (4)$$

$\Delta$  is a homomorphism so  $m_{H \otimes H} \circ (\Delta \otimes \Delta) = \Delta \circ m$ . (4) can therefore be transferred to the formula

$$H \otimes H \xrightarrow{m} H \xrightarrow{\Delta} H \otimes H \xrightarrow{s \otimes \text{Id}} H \otimes H \xrightarrow{m} H, \quad (5)$$

and this to  $H \otimes H \xrightarrow{m} H \xrightarrow{1_H} H = \mathbf{1}_{H \otimes H, H}$  because  $s$  is an antipode.

$m \star (m \circ (s \otimes s) \circ \tau) : H \otimes H \xrightarrow{\Delta_{H \otimes H}} H \otimes H \otimes H \otimes H \xrightarrow{m \circ (m \circ (s \otimes s) \circ \tau)} H \otimes H \xrightarrow{m} H$ .  
Now  $\Delta_{H \otimes H} = (\text{Id}_H \otimes \tau \otimes \text{Id}_H) \circ (\Delta \otimes \Delta)$  and  $m \circ (m \circ (s \otimes s) \circ \tau) = (m \otimes m) \circ (\text{Id}_H \otimes \text{Id}_H \otimes s \otimes s) \circ (\text{Id}_H \otimes \text{Id}_H \otimes \tau)$ . So  $m \star (m \circ (s \otimes s) \circ \tau)$  can be split up into

$$H \otimes H \xrightarrow{\Delta \otimes \Delta} H \otimes H \otimes H \otimes H \xrightarrow{\text{Id} \otimes \tau \otimes \text{Id}} H \otimes H \otimes H \otimes H \xrightarrow{\text{Id} \otimes \text{Id} \otimes \tau}$$

$H \otimes H \otimes H \otimes H$  and

$$H \otimes H \otimes H \otimes H \xrightarrow{\text{Id} \otimes \text{Id} \otimes s \otimes s} H \otimes H \otimes H \otimes H \xrightarrow{m \otimes m} H \otimes H \xrightarrow{m} H.$$

To be able to trace the capital aitches we prime one of them. The former of these two mappings becomes in this way

$$H \otimes H' \xrightarrow{\Delta \otimes \Delta} H \otimes H \otimes H' \otimes H' \xrightarrow{\text{Id} \otimes \tau \otimes \text{Id}} H \otimes H' \otimes H \otimes H' \xrightarrow{\text{Id} \otimes \text{Id} \otimes \tau} H \otimes H' \otimes H' \otimes H.$$

$m \circ (m \otimes m) = H \otimes H' \otimes H' \otimes H \xrightarrow{\text{Id} \otimes m \otimes \text{Id}} H \otimes H' \otimes H \xrightarrow{m \otimes \text{Id}} H \otimes H \xrightarrow{m} H$ . So, the mapping  $H' \xrightarrow{\Delta} H' \otimes H' \xrightarrow{\text{Id} \otimes s} H' \otimes H' \xrightarrow{m} H'$  is hidden in  $m \star (m \circ (s \otimes s) \circ \tau)$  and

$H' \xrightarrow{\Delta} H' \otimes H' \xrightarrow{\text{Id} \otimes s} H' \otimes H' \xrightarrow{m} H' = \mathbf{1}_H$ . Of  $m \star (m \circ (s \otimes s) \circ \tau)$  there remains  $H \xrightarrow{\Delta} H \otimes H \xrightarrow{\text{Id} \otimes s} H \otimes H \xrightarrow{m} H$  because the images of  $\mathbf{1}_H$  are scalars. So  $m \star (m \circ (s \otimes s) \circ \tau) = \mathbf{1}_{H \otimes H, H}$ .  $m$  therefore has a left and a right inverse so these inverses must be equal.

2. The dual of formula (1) is

$$\Delta \circ s = \tau \circ (s \otimes s) \circ \Delta. \quad (6)$$

We shall prove this by showing that  $\Delta \circ s$  is a left and that  $\tau \circ (s \otimes s) \circ \Delta$  is a right inverse of  $\Delta$ .

$$\begin{aligned} (\Delta \circ s) \star \Delta &= H \xrightarrow{\Delta} H \otimes H \xrightarrow{(\Delta \circ s) \otimes \Delta} H \otimes H \otimes H \otimes H \xrightarrow{m_{H \otimes H}} H \otimes H = \\ &H \xrightarrow{\Delta} H \otimes H \xrightarrow{s \otimes \text{Id}} H \otimes H \xrightarrow{\Delta \otimes \Delta} H \otimes H \otimes H \otimes H \xrightarrow{m_{H \otimes H}} H \otimes H = \\ &H \xrightarrow{\Delta} H \otimes H \xrightarrow{s \otimes \text{Id}} H \otimes H \xrightarrow{m} H \xrightarrow{\Delta} H \otimes H = \mathbf{1}_{H, H \otimes H}. \end{aligned}$$

$$\begin{aligned} \Delta \star (\tau \circ (s \otimes s) \circ \Delta) &= H \xrightarrow{\Delta} H \otimes H \xrightarrow{\Delta \otimes \Delta} H \otimes H' \otimes H' \otimes H \xrightarrow{\text{Id} \otimes \text{Id} \otimes s \otimes s} \\ &H \otimes H' \otimes H' \otimes H \xrightarrow{\text{Id} \otimes \text{Id} \otimes \tau} H \otimes H' \otimes H \otimes H' \xrightarrow{\text{Id} \otimes \tau \otimes \text{Id}} H \otimes H \otimes H' \otimes H' \xrightarrow{m \otimes m} \\ &H \otimes H' \text{ because } H \otimes H' \otimes H \otimes H' \xrightarrow{\text{Id} \otimes \tau \otimes \text{Id}} H \otimes H \otimes H' \otimes H' \xrightarrow{m \otimes m} \\ &H \otimes H' = m_{H \otimes H}. \end{aligned}$$

$H$  is coassociative and this gives the following commutative diagram.

$$\begin{array}{ccc} H & & \\ \Delta \downarrow & & \\ H \otimes H & \xrightarrow[\text{Id} \otimes \Delta]{\Delta \otimes \text{Id}} & H \otimes H' \otimes H \\ \Delta \otimes \text{Id} \downarrow \quad \text{Id} \otimes \Delta \downarrow & & \downarrow \text{Id} \otimes \Delta \otimes \text{Id} \\ H \otimes H \otimes H & \xrightarrow{\text{Id} \otimes \text{Id} \otimes \Delta} & H \otimes H' \otimes H' \otimes H \end{array}$$

It is clear that  $H' \otimes H'$  stems from  $H'$  but the two unprimed aitches of  $H \otimes H' \otimes H' \otimes H$  are derived from  $H$  because according to the diagram first  $\Delta$  is applied and then two times  $\text{Id}_H$ . Therefore  $H' \xrightarrow{\Delta} H' \otimes H' \xrightarrow{\text{Id} \otimes s} H' \otimes H' \xrightarrow{m} H'$  is hidden in  $\Delta \star (\tau \circ (s \otimes s) \circ \Delta)$  and what remains is the same application but then without primes. The left inverse of  $\Delta$  is equal to the right inverse.

The *opponent*  $H^\circ$  of  $H$  is the bialgebra which is as set equal to  $H$  and which has the multiplication  $m^\circ = m \circ \tau$  and the coproduct  $\Delta^\circ = \tau \circ \Delta$ .

The formulae (1) and (6) are equivalent with the commutativity of the next two diagrams.

$$\begin{array}{ccc}
\begin{array}{ccc}
H & \xleftarrow{m} & H \otimes H \\
\downarrow s & & \downarrow s \otimes s \\
H^\circ & \xleftarrow[m^\circ]{m \circ \tau} & H^\circ \otimes H^\circ
\end{array} & \text{and} & 
\begin{array}{ccc}
H & \xrightarrow{\Delta} & H \otimes H \\
\downarrow s & & \downarrow s \otimes s \\
H^\circ & \xrightarrow[\Delta^\circ]{\tau \circ \Delta} & H^\circ \otimes H^\circ
\end{array}
\end{array}$$

3. If we prove that  $s(1) = 1$  and that  $\epsilon \circ s = \epsilon$  then it follows that  $s$  is an isomorphism of  $H$  onto  $H^\circ$ .

$(s \star \text{Id}_H)(1) = 1$  and  $= s(1)$ . The second relation is more difficult. If  $f: H \rightarrow V$  is a linear mapping then

$$f = H \xrightarrow{\Delta} H \otimes H \xrightarrow{f \otimes \epsilon} V \otimes K \xrightarrow{m} V.$$

This follows from the equality

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{\text{Id} \otimes \epsilon} H \otimes K = (h \rightarrow h \otimes 1 (h \in H)).$$

Therefore  $s = H \xrightarrow{\Delta} H \otimes H \xrightarrow{s \otimes \epsilon} H \otimes K \xrightarrow{m} H$  and this can be written as

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{s \otimes 1} H \otimes H \xrightarrow{m} H. \text{ So, } 1 \circ s \text{ (where } 1 \text{ is the unit of } \mathcal{L}(H)) =$$

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{s \otimes 1} H \otimes H \xrightarrow{m} H \xrightarrow{1} H =$$

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{s \otimes 1} H \otimes H \xrightarrow{1 \otimes 1} H \otimes H \xrightarrow{m} H =$$

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{s \otimes \text{Id}} H \otimes H \xrightarrow{1 \otimes 1} H \otimes H \xrightarrow{m} H =$$

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{s \otimes \text{Id}} H \otimes H \xrightarrow{m} H \xrightarrow{1} H = 1 \circ 1 = 1. \text{ It follows that } \epsilon \circ s = \epsilon.$$

4. The last property of  $s$  that we prove is  $s \circ s = \text{Id}_H$ . In other words we show that  $s$  is an *involution*.

$$s \star (s \circ s) = H \xrightarrow{\Delta} H \otimes H \xrightarrow{s \otimes (s \circ s)} H \otimes H \xrightarrow{m} H =$$

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{s \otimes s} H \otimes H \xrightarrow{\text{Id} \otimes s} H \otimes H \xrightarrow{m} H =$$

$$H \xrightarrow{s} H \xrightarrow{\Delta} H \otimes H \xrightarrow{\tau} H \otimes H \xrightarrow{\text{Id} \otimes s} H \otimes H \xrightarrow{m} H$$

because  $s$  is a coalgebramorphism.

$$\text{So, } s \star (s \circ s) = H \xrightarrow{s} H \xrightarrow{\Delta} H \otimes H \xrightarrow{s \otimes \text{Id}} H \otimes H \xrightarrow{m} H = 1 \circ s = 1.$$

DEFINITION. A linear space  $H$  over a field  $K$  is a **Hopf algebra** if  $H$  is a bialgebra and possesses an antipode.

## 2. EXAMPLES

### 1. Group algebras.

Let  $G$  be a group and  $K$  a field. The linear space  $H = K^{(G)}$  consists of finite linear combinations of group elements, i.e.  $x \in H$  means that there is a family  $(x_g)_{g \in G}$  of  $K$  with  $x_g \neq 0$  for at most a finite number of  $g \in G$  and that  $x = \sum_{g \in G} x_g e_g$  where  $e_g$  is an

alternative notation for  $g$ .  $(e_g)$  is the canonical base of the linear space  $H$ . We define a multiplication  $m$  as the linear mapping  $H \otimes H \rightarrow H$  with  $m(e_g \otimes e_h) = e_{gh}$ .  $H$  is with this  $m$  an associative algebra with unit  $e_1$  ( $1$  is the unit of  $G$ ). We define the coproduct  $\Delta$  as the linear mapping  $H \rightarrow H \otimes H$  with  $\Delta(e_g) = e_g \otimes e_g$ .  $\Delta$  is a homomorphism:  $\Delta(e_g e_h) = \Delta(e_{gh}) = e_{gh} \otimes e_{gh} = (e_g \otimes e_g)(e_h \otimes e_h)$ .  $H$  has the counit  $\epsilon$  with  $\epsilon(e_g) = 1$  for all  $g \in G$ .  $\epsilon$  is a homomorphism too.  $H$  is in this way a bialgebra. There is an antipode  $s$ : a linear endomorphism of  $H$  with  $s(e_g) = e_{g^{-1}}$ .

This example has caused the following definition: if  $H$  is a Hopf algebra with coproduct  $\Delta$  then  $x \in H$  is grouplike if  $\Delta(x) = x \otimes x$ .

## 2. Enveloping and symmetric algebras of Lie algebras.

Let  $V$  be a vectorspace over a field  $K$ . We can construct the *tensoralgebra*  $T(V)$  as follows:  $T(V) = K \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$ .  $T(V)$  has an  $N$ -grading:  $T^n = V \otimes \dots \otimes V$  ( $n$  factors),  $T^0 = K$ .  $T(V)$  has also a  $N$ -filtering:  $T_n = \bigoplus_{m=0}^n T^m$ . It follows that  $T_0 = K$  and that  $T_m \otimes T_n \subset T_{m+n}$  (the multiplication of  $T(V)$  is given by  $\otimes$  itself).

$\mathfrak{g}$  is a Lie algebra with underlying space  $V$ . The envelope  $U(\mathfrak{g})$  is the quotient of  $T(\mathfrak{g})$  and the two-sided ideal  $J$  which is generated by elements of the shape  $x \otimes y - y \otimes x - [x, y]$  with  $x$  and  $y \in \mathfrak{g}$ .  $U$  has an  $N$ -filtering:  $U_n$  is the image of  $T_n$  under the canonical mapping  $T \rightarrow U$ . It follows that  $U_0 = K$ , that  $U_n \subset U_{n+1}$  and that  $U_m U_n \subset U_{m+n}$ .

Let  $G^n = U_n/U_{n-1}$  if  $n > 0$ ,  $G^0 = U_0 = K$  and  $G = \bigoplus_{n \geq 0} G^n$ . For  $\xi \in G^m$

and  $\eta \in G^n$  we define the product  $\xi\eta = xy + U_{m+n-1}$  if  $\xi = x + U_{m-1}$  and if  $\eta = y + U_{n-1}$ . This product is well defined because if we take  $x' \in U_m$  and  $y' \in U_n$  such that  $x' - x \in U_{m-1}$  and  $y' - y \in U_{n-1}$  then  $\xi\eta = x'y' + U_{m+n-1}$ . In this way  $G$  becomes an associative algebra with the  $1$  of  $K$  as unit. The element  $x \otimes y - y \otimes x$  has degree 2 and the element  $[x, y]$  degree 1 whereas their images in  $U$  are equal. That means that  $G$  is *commutative* and this fact gives rise to the following construction.

### Poisson bracket.

For  $x \in U_m$  and  $y \in U_n$ ,  $xy - yx \in U_{m+n-1}$  because  $\xi\eta - \eta\xi = 0$  with  $\xi = x + U_{m-1}$  and  $\eta = y + U_{n-1}$ . So, for  $\xi$  and  $\eta$  we can define  $\{\xi, \eta\} = xy - yx + U_{m+n-2}$ . This is well defined for if  $\xi = x' + U_{m-1}$  and  $\eta = y' + U_{n-1}$  than

$$\begin{aligned} xy - yx - x'y' + y'x' &= (x - x')(y - y') + x'(y - y') + (y - y')x' - \\ &= (y - y')(x - x') - y'(x - x') - (y - y')x' = \\ &= (x - x')(y - y') - (y - y')(x - x') + \end{aligned} \tag{i}$$

$$x'(y - y') - (y - y')x' + \tag{ii}$$

$$(x - x')y' - y'(x - x'). \tag{iii}$$

(i), (ii) and (iii)  $\in U_{m+n-2}$ .

$G$  is with this bracket a Lie algebra but  $\{ , \}$  has an extra property:  
 If  $\alpha = a + U_{p-1}$  then  $\{\alpha, \xi\eta\} =$   
 $axy - xy a + U_{m+n+p-2} = (ax - xa)y + x(ay - ya) + U_{m+n+p-2} =$   
 $\{\alpha, \xi\}\eta + \xi\{\alpha, \eta\}$ . So,  $\{ , \}$  is a Poisson bracket.

It follows from the theorem of Poincaré, Birkhoff and Witt that the restriction of the canonical mapping  $T \rightarrow U$  to  $T^1 = \mathfrak{g} = V$  is *injective*. So we can view upon  $\mathfrak{g}$  as embedded in  $U$ .  $U$  has a coproduct  $\Delta$  which is a homomorphic extension to  $U$  of the linear application  
 $x \mapsto x \otimes 1 + 1 \otimes x (x \in \mathfrak{g})$ .

This homomorphism is possible because  $\Delta(xy - yx) =$   
 $(xy - yx) \otimes 1 + 1 \otimes (xy - yx)$  while  $\Delta([x, y]) = [x, y] \otimes 1 + 1 \otimes [x, y]$  and these relations are consistent with each other. If  $U$  has a counit  $\epsilon$  then there must hold:  $(\epsilon \otimes \text{Id}_U) \circ \Delta(x) = h(x)$  or  $\epsilon(x) \otimes 1_U + \epsilon(1_U) \otimes x = 1_K \otimes x$  for  $x \in \mathfrak{g}$ . It follows that  $\epsilon(x) = 0$  and  $\epsilon(1_U) = 1_K$ . So,  $\epsilon$  applies every element of  $U$  to its constant part (i.e. the term which contains no elements of  $\mathfrak{g}$ ).  $\epsilon$  is also a homomorphism. If there exists an antipode  $s$  then there must hold:  $m \circ (s \otimes \text{Id}_U) \circ \Delta(x) = 1_U(x)$  for all  $x \in U$  and  $s(1) = 1$ . For  $x \in \mathfrak{g}$  it follows that  $s(x) + x = 0$ . If we take an element  $y \in \mathfrak{g}$  then  $s([x, y]) = -[x, y]$  must hold. On the other hand  $s$  must be a homomorphism of  $U$  in  $U^\circ$  so that  $s(xy - yx)$  must be equal to  $(-y)(-x) - (-x)(-y)$  and this  $= -[x, y]$  in  $U$ . So,  $s$  exists. Kostant proved in the 1950's that all cocommutative Hopf algebras over a field with characteristic 0 which have an  $N$ -filtering as above are envelopes of a Lie algebra the elements of which are characterized by the formula  $\Delta(x) = x \otimes 1 + 1 \otimes x$ . These elements are called *primitive*.

The underlying space  $V$  of  $\mathfrak{g}$  can also be viewed upon as a Lie algebra with  $[x, y] = 0$  for all  $x$  and  $y \in V$ . The envelope of this algebra is the so-called *symmetric algebra*  $S(\mathfrak{g})$ .  $S$  is commutative and can be identified to  $K[e_\lambda]_{\lambda \in \Lambda}$  where  $(e_\lambda)$  is a base of  $V$ . So the elements of  $S$  are polynomials in the variables  $e_\lambda$ .

It is relatively easy to construct out of the canonical mappings  $T \rightarrow U$ ,  $T \rightarrow S$  and  $U \rightarrow G$  a homomorphism of  $S$  onto  $G$  (also yielding the commutativity of  $G$ ). Poincaré, Birkhoff and Witt proved that this homomorphism is also *one to one*. From now on we identify  $G$  and  $S$  so that the Poisson bracket is defined on  $S$ .

From the fact that  $G = S$  follows that there is also an *isomorphism* of  $S$  into  $U$ , not an algebraic one of course but a *coalgebraic* one. This isomorphism is given by the formula

$$\eta(x_1 \dots x_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} x_{\sigma(1)} \dots x_{\sigma(n)} \quad (7)$$

where  $\mathfrak{S}_n$  is the permutation group of  $n$  elements,  $x_1 \dots x_n$  being elements of  $\mathfrak{g}$ . Especially  $\eta(x) = x$  for  $x \in \mathfrak{g}$ , so,  $\eta$  has an interesting impact on  $\{ , \}$ :  $\eta(\{x, y\}) = [x, y] = xy - yx$  for all  $x$  and  $y \in \mathfrak{g}$ . The formula still holds, i.e.  $\eta(\{x, y\}) = [\eta(x), \eta(y)]$ , if *one* of them is element of  $S$  and the other of  $\mathfrak{g}$ .

*The case of finite dimensional Lie algebras.*

Let  $\mathfrak{g}$  has the base  $x_1, \dots, x_n$  then  $S = K[x_1, \dots, x_n]$ . With induction it follows immediately that  $\{x, y_i^k\} = ky_i^{k-1}\{x, y_i\}$  for  $x \in S$  and  $y_i \in \mathfrak{g}$ . From this it follows easily that  $\{x, g\} = \sum_{i=1}^n \frac{\partial g}{\partial x_i} \{x, x_i\}$  if  $g \in S$  and that  $\{f, g\} = \sum_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} [x_i, x_j]$  if  $f \in S$  too.

Let now  $(q_1, \dots, q_n, p_1, \dots, p_n)$  be a base of a commutative Lie algebra  $\mathfrak{g}$  and  $\Phi : \mathfrak{g} \times \mathfrak{g} \rightarrow K$  bilinear and alternating with  $\Phi(q_i, q_j) = \Phi(p_i, p_j) = 0$  and with  $\Phi(q_i, p_j) = \delta_{ij}$  then  $\Phi$  (trivially) satisfies the relation  $-\Phi([x, y], z) + \Phi([x, z], y) - \Phi([y, z], x) = 0$ , in other words  $\Phi$  is a *cocycle*. The linear space  $\mathfrak{h} = \mathfrak{g} \times K$  is a Lie algebra with

$$[(x, \alpha), (y, \beta)] = ([x, y], \Phi(x, y))$$

for  $x$  and  $y \in \mathfrak{g}$  and  $\alpha$  and  $\beta \in K$ . We get  $\mathfrak{h} = (\mathbf{q}_1, \dots, \mathbf{q}_n, \mathbf{p}_1, \dots, \mathbf{p}_n, \omega)$  with  $\mathbf{q}_i = (q_i, 0)$ ,  $\mathbf{p}_i = (p_i, 0)$  and  $\omega = (0, 1)$ .  $\mathfrak{h}$  is a *Heisenberg algebra* and on  $S(\mathfrak{h})$  the Poisson bracket is given by

$$\{f, g\} = \sum_{i=1}^n \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \omega. \quad (8)$$

It is worthwhile to compare this relationship between  $S$  and  $U$  with section 23 of the sixth chapter of the book "Quantum Mechanics" by L.I. Schiff. This relationship is also probably the code to decipher the cryptic book "Operators" by V.P. Maslow.

### 3. Compact groups and Hopf algebras.

Let  $G$  be a compact group and  $\mathcal{C}(G) = H$  the set of continuous functions  $G \rightarrow \mathbf{R}$ . With the multiplication  $(fg)(x) = f(x)g(x)$  the space  $H$  is a *commutative algebra* with unit 1 (the function  $x \mapsto 1(x \in G)$ ).  $H$  has the norm  $\| \cdot \|$  given by  $\|f\| = \max_{x \in G} \|f(x)\|$ .  $H$  is a Banach space, moreover the inequality  $\|fg\| \leq \|f\| \|g\|$  holds. For  $f$  and  $g \in H$  we may identify  $f \otimes g$  with the function  $(x, y) \mapsto f(x)g(y)$  ( $x, y \in G$ ) as is well known. According to the theorem of Weierstraß and Stone  $H \otimes H$  is dense in  $\mathcal{C}(G \times G)$  so that this set is the topological tensor product  $H \hat{\otimes} H$ . We define a coproduct  $\Delta$  as follows:

$$\Delta(f)(x, y) = f(xy).$$

$G$  is associative so  $\Delta$  is coassociative. One can see that by a close look to the applications

$$H \otimes H \xrightarrow{\text{Id} \otimes \Delta} H \otimes H \hat{\otimes} H \text{ and } H \otimes H \xrightarrow{\Delta \otimes \text{Id}} H \hat{\otimes} H \otimes H.$$

$\Delta$  is cocommutative if  $G$  is commutative. There is a counit  $\epsilon$  given by

$$\epsilon(f) = f(e) \quad (e \text{ is the unit of } G).$$

To see that  $\epsilon$  is a correct counit one could look first at the mappings

$$H \otimes H \xrightarrow{\epsilon \otimes \text{Id}} \mathbf{R} \otimes H = H \text{ and } H \otimes H \xrightarrow{\text{Id} \otimes \epsilon} H \otimes \mathbf{R} = H.$$

With these compositions  $H$  becomes a bialgebra. There is also an antipode  $s$  given by

$$s(f)(x) = f(x^{-1}).$$

We prove this as follows. First we must extend the multiplication to a continuous linear mapping  $m : H \hat{\otimes} H \rightarrow H$ . Let  $f \in H \hat{\otimes} H$  then there is a sequence  $(g_n \otimes h_n)$  on  $H \otimes H$  with converges to  $f$ . That means that  $g_n(x)h_n(y)$  converges uniformly in  $x$  and  $y$  to  $f(x, y)$ . Now  $m(g_n \otimes h_n)$  is the function  $x \mapsto g_n(x)h_n(x)$  so  $m(f)$  is the function  $x \mapsto f(x, x)$ .

If  $f \in H$  and if  $s$  exists then there must hold  $m \circ (s \otimes \text{Id}) \circ \Delta(f) = \mathbf{1}_H(f)$ .

Let the sequence  $(g_n \otimes h_n)$  on  $H \otimes H$  converge to  $\Delta(f)$  then

$\lim_{n \rightarrow \infty} g_n(x)h_n(y) = f(xy)$  for all  $x$  and  $y \in G$ . If  $s(g_n) = g'_n$  then  $\lim_{n \rightarrow \infty} g'_n(x)h_n(y)$  exists because  $s$  is assumed to be continuous.

$m(g'_n \otimes h_n)(x) = g'_n(x)h_n(x)$  and this must converge to  $\mathbf{1}_H(f)(x) = f(e)1 = f(e)$ . For  $g'_n(x) = g_n(x^{-1})$  all our demands can be met with.  $s$  is determined because  $\text{Id}$  can have only one inverse.

Let  $\mathcal{M} = H'$  be the topological dual of  $H$ .  $\mathcal{M}$  is the set of *measures* on  $G$ . We shall prove that  $\mathcal{M}$  is a cocommutative Hopf algebra.

$\mathcal{M}$  is an associative algebra with  $m'(\lambda \otimes \mu) = \lambda \star \mu$ , the already known convolution of the elements  $\lambda$  and  $\mu \in \mathcal{M}$ . In the language of measure theory we have the formula

$$(\lambda \star \mu)(f) = \int_{G \times G} f(xy) d\lambda(x) d\mu(y).$$

The counit  $\epsilon$  of  $H$  is the unit of  $\mathcal{M}$ .

The norm on  $\mathcal{M}$  is given by  $\|\lambda\| = \sup_{f \neq 0} \frac{|\lambda(f)|}{\|f\|}$ . The real combinations of the

Dirac measures  $\epsilon_x$  form a dense subalgebra ( $\epsilon_x(f) = f(x)$ ,  $\epsilon_{xy} = \epsilon_x \star \epsilon_y$ )

which is nothing else than the group algebra  $\mathbf{R}^{(G)}$ .

$m'$  is the transpose of  $\Delta : \langle m'(\lambda \otimes \mu), f \rangle = (\lambda * \mu)(f) = m_{\mathbf{R}} \circ (\lambda \otimes \mu) \circ \Delta(f) = \langle \lambda \otimes \mu, \Delta(f) \rangle$ ,  $m_{\mathbf{R}}$  being the multiplication of  $\mathbf{R}$ .

The transpose  $\Delta'$  of  $m$  is the coproduct of  $\mathcal{M}$ :

for  $f \in H \hat{\otimes} H = \mathcal{C}(G \times G)$  we have  $\langle \Delta'(\lambda), f \rangle = \langle \lambda, m(f) \rangle = \int_G f(x, x) d\lambda(x)$ .

$\Delta'$  is a homomorphism for on the one hand we have

$\langle \Delta'(\lambda * \mu), f \rangle = \int_G f(x, x) d(\lambda * \mu)(x) = \int_{G \times G} f(xy, xy) d\lambda(x) d\mu(y)$  and on the other hand

$\langle \Delta'(\lambda) * \Delta'(\mu), f \rangle = \int_{G \times G \times G \times G} f((x, x')(y, y')) d\Delta'(\lambda)(x, x') d\Delta'(\mu)(y, y') = \int_{G \times G \times G \times G} f(xy, x'y') d\Delta'(\lambda)(x, x') d\Delta'(\mu)(y, y') = \int_{G \times G} f(xy, xy) d\lambda(x) d\mu(y)$ , because  $\int_{G \times G} f(x, y) d\Delta'(\lambda)(x, y) = \int_G f(x, x) d\lambda(x)$ .

For  $\lambda = \epsilon_x$  it follows that  $\Delta'(\epsilon_x) = \epsilon_x \otimes \epsilon_x$  because  $\langle \Delta'(\epsilon_x), g \otimes h \rangle = \langle \epsilon_x, m(g \otimes h) \rangle = g(x)h(x) = \epsilon_x(g)(\epsilon_x(h)) = (\epsilon_x \otimes \epsilon_x)(g \otimes h)$  and because  $H \otimes H$  is dense in  $\mathcal{C}(G \times G)$ . So,  $G$  yields the grouplike elements of  $\mathcal{M}$ .  $\mathcal{M}$  is cocommutative because  $H$  is commutative and because the transpose  $\tau'$  of  $\tau$  is the application  $\lambda \otimes \mu \mapsto \mu \otimes \lambda$ .

The counit  $\epsilon'$  is given by  $\epsilon'(\lambda) = \langle \lambda, 1 \rangle = \int_G d\lambda(x)$ .  $\epsilon'$  is the transpose of  $i$ , the embedding of  $\mathbf{R}$  into  $H$ .  $\epsilon'(\lambda * \mu) = \int_{G \times G} d\lambda(x) d\mu(y) = \epsilon'(\lambda)\epsilon'(\mu)$ .

The antipode  $s'$  is given by  $s'(\lambda)(f) = \int_G f(x^{-1}) d\lambda(x)$ , so  $s'$  is the transpose of  $s$ . The embedding of  $\mathbf{R}$  into  $\mathcal{M}$  is the transpose of  $\epsilon$  and  $\epsilon'$  is the transpose of the embedding of  $\mathbf{R}$  into  $H$  so  $1_{\mathcal{M}}$  is the transpose of  $1_H$ . It follows that  $s'$  satisfies the definition of antipode because all appearing mappings are transposes of  $\Delta, s, \text{Id}$  and  $m$ . It goes without saying that all applications involved are continuous.

#### 4. Alternating and Clifford algebras.

$V$  is again a vectorspace over a field  $K$  with characteristic 0.  $Q$  is a quadratic form on  $V$ :  $Q(\alpha x) = \alpha^2 Q(x)$  for  $\alpha \in K$  and  $x \in V$ ;

$Q(x + y) - Q(x) - Q(y)$  is a bilinear form  $\Phi(x, y)$  on  $V$ .  $Q$  is called *degenerate* if  $\Phi$  is degenerate.  $C(Q)$  is the algebra over  $K$  equal to the quotient of  $T(V)$  and the two-sided ideal  $I(Q)$  generated by expressions of the form  $x \otimes x - Q(x)$ .  $K$  and  $V$  are embedded into  $C(Q)$  and in  $C(Q)$ ,  $x^2$  equals  $Q(x)$  for  $x \in V$ .  $C(Q)$  is associative and possesses a unit 1.  $C(Q)$  is called *Clifford algebra of the quadratic space*  $(V, Q)$ . If  $Q = 0$  for all  $x \in V$  then  $C(Q)$  is nothing else than the *exterior algebra*  $\Lambda(V)$  of  $V$ .  $C(Q)$  is characterized by the following property: *For any linear mapping  $f$  of  $V$  into an associative algebra  $A$  with unit  $1_A$  with the property that  $f(x)^2 = Q(x)1_A$  there is one and only one homomorphism  $\bar{f} : C(Q) \rightarrow A$  which is an extension of  $f$ . ( $\bar{f}$  is the so-called lift of  $f$ ).*



$C(Q)$  has a *grading*  $C^+ \oplus C^-$ , the elements of  $C^+$  are called *even* (degree 0) and those of  $C^-$  are called *uneven* (degree 1).  $C^+$  is a subalgebra of  $C(Q)$  ( $1 \in C^+$ ) and  $C^-$  is a  $C^+$ -module, moreover  $C^-C^- \subset C^+$ .

Let  $(x_i)_{i \in L}$  be a base of  $V$  where  $L$  is totally ordered. For a finite subset  $H = (h_1, \dots, h_q)$  ( $h_1 < \dots < h_q$ ) of  $L$  we write  $x_H$  for  $x_{h_1} \dots x_{h_q}$ . The  $x_H$  form a base of the linear space  $C(Q)$ . There are linear isomorphisms  $\lambda_Q : C(Q) \rightarrow C(2Q)$  and  $\mu_Q : C(Q) \rightarrow \Lambda(V)$  with  $\lambda_Q(x_H) = x_H$  and  $\mu_Q(x_H) = x_H$ . The inverse of  $\mu_Q$  shall be denoted by  $\eta_Q$ . Here  $C(2Q)$  is the algebra of the quadratic space  $(V, 2Q)$ . In some cases there is an *algebraic* isomorphism  $C(Q) \rightarrow C(2Q)$ . We shall not use this fact.

Let  $(V_i, Q_i)$  ( $i = 1, 2$ ) be two quadratic spaces with Clifford algebras  $C(Q_i)$ . Let  $V = V_1 \oplus V_2$  and define  $Q$  on  $V$  by putting  $Q(x_1 + x_2) = Q_1(x_1) + Q_2(x_2)$  ( $x_i \in V_i$ ) then  $(V, Q)$  is a quadratic space too and the  $V_i$  are orthogonal to each other. This means that if  $\Phi(x, y) = Q(x + y) - Q(x) - Q(y)$  that then  $\Phi(x, y) = 0$  for  $x \in V_1$  and  $y \in V_2$ . The Clifford algebra  $C(Q)$  is *algebraically isomorphic* to the algebra  $C(Q_1) \otimes C(Q_2)$ , the isomorphism being given by the lift of  $x_1 + x_2 \mapsto x_1 \otimes 1 + 1 \otimes x_2$  for  $x_i \in V_i$ . The multiplication of  $C(Q_1) \otimes C(Q_2)$  should be given by  $(a_1 \otimes a_2)(b_1 \otimes b_2) = \epsilon(a_1 b_1) \otimes (a_2 b_2)$  with  $a_i$  and  $b_i$  even or uneven elements of  $C(Q_1)$  and  $C(Q_2)$  respectively.  $\epsilon = -1$  if both  $a_2$  and  $b_1$  are uneven,  $\epsilon = 1$  in all other cases.

The linear mapping  $x \mapsto x \otimes 1 + 1 \otimes x$  from  $V$  into  $C(Q) \otimes C(Q)$  can be lifted to a homomorphism  $j : C(2Q) \rightarrow C(Q) \otimes C(Q)$  because  $(x \otimes 1 + 1 \otimes x)^2 = 2Q(x)(1 \otimes 1)$ , so  $\Delta_Q = j \circ \lambda_Q$  is a linear mapping  $C(Q) \rightarrow C(Q) \otimes C(Q)$ . If  $Q = 0$  then  $\Delta_Q$  is a homomorphism because  $\lambda_0$  is the identity mapping. We shall show that  $C(Q)$  is a coassociative coalgebra with counit  $\epsilon_Q$  and with  $\Delta_Q$  as a coproduct. We use our base  $(x_H)$  again.  $\lambda_Q(x_H) = x_H$  so we can act as if  $\Delta_Q$  is the homomorphism  $j$ . It is then sufficient to calculate  $(\Delta_Q \otimes \text{Id}) \circ \Delta_Q(x)$  and  $(\text{Id} \otimes \Delta_Q) \circ \Delta_Q(x)$  for  $x \in V$ . The result is two times  $x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x$ . So,  $C(Q)$  is coassociative. If we put  $\epsilon_Q(x_H) = 0$  if  $H \neq \emptyset$  and  $= 1$  if  $H = \emptyset$  ( $1 = x_H$  if  $H = \emptyset$ ) then  $\epsilon_Q$  is the counit according to the definition of a counit. The proof of this is the same as that of the coassociativity.

We shall now show that  $C(Q)$  is *coanticommutative*, i.e. that  $\sigma \circ \Delta_Q = \Delta_Q \circ \sigma$ ,  $\sigma$  being the linear mapping with  $\sigma(a \otimes b) = \epsilon(b \otimes a)$  where  $a$  and  $b$  are even or uneven and  $\epsilon = -1$  if both are uneven,  $= 1$  in the other cases. Let  $x_1, \dots, x_n$  be elements of the chosen base of  $V$  then  $\lambda_Q(x_1 \dots x_n) = x_1 \dots x_n$  so that

$$\Delta_Q(x_1 \dots x_n) = (x_1 \otimes 1 + 1 \otimes x_1) \dots (x_n \otimes 1 + 1 \otimes x_n) =$$

$$\sum_{i=0}^n \sum_{\alpha \in I(i)} \epsilon_{\alpha}(x_{\alpha(1)} \dots x_{\alpha(i)}) \otimes (x_{\alpha(i+1)} \dots x_{\alpha(n)}), \quad (9)$$

$I(i) = \{\alpha \in \mathfrak{G}_n \mid \alpha \text{ increasing on both intervals } [1, i] \text{ and } [i+1, n] \text{ of } N\}$  and  $\epsilon_{\alpha}$  is the sign of the permutation  $\alpha$ . The factors  $x_{\alpha(1)} \dots x_{\alpha(i)}$  and  $x_{\alpha(i+1)} \dots x_{\alpha(n)}$  are even or uneven because the  $x_i$  are elements of  $V$  and so uneven. It follows that  $\sigma$  just interchanges the terms of  $\Delta_Q(x_1 \dots x_n)$ . This operation applies to all the  $x_H$ .

In particular  $\Lambda(V)$  is a coassociative, coanticommutative coalgebra with a counit. Moreover,  $\Delta_0$  and  $\epsilon_0$  are homomorphisms, so that  $\Lambda(V)$  is a *bialgebra*.

We shall now show that  $\eta_Q$  is a coalgebra isomorphism of  $\Lambda(V)$  onto  $C(Q)$ . If  $x_1, \dots, x_n$  are elements of the chosen base of  $V$  then  $\eta_Q(x_1 \wedge \dots \wedge x_n) = x_1 \dots x_n$ . It follows from (9) that  $(\eta_Q \otimes \eta_Q) \circ \Delta_Q(x_H) = \Delta_Q \circ \eta(x_H)$ . One can prove that if  $(y_1, \dots, y_n)$  is any family of elements of  $V$  that the following formula holds:

$$\eta_Q(y_1 \wedge \dots \wedge y_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{G}_n} \epsilon_{\sigma} y_{\sigma(1)} \dots y_{\sigma(n)}. \quad (10)$$

Although  $C(Q)$  is in general not a bialgebra it possesses an automorphism that plays the role of antipode. This automorphism  $\alpha$  is given by  $\alpha(x) = -x$  for  $x \in V$ .  $\alpha$  is the identity mapping on  $C^+$  and  $\alpha(u) = -u$  if  $u \in C^-$ .  $\alpha$  is also a coalgebra morphism. On  $\Lambda(V)$ , which is a bialgebra,  $\alpha$  is an antipode. One should keep in mind here that  $\Lambda(V)$  is equal to its own opponent with respect to  $\sigma$ .  $\alpha$  is called *principal automorphism*. There is also a *principal antiautomorphism*  $\beta$ . It is the isomorphism of  $C(Q)$  onto the opponent of  $C(Q)$  with respect to  $\tau$  that extends the mapping  $x \mapsto x(x \in V)$ .

#### *Poisson bracket.*

Let  $C_h$  be the image of  $T_h$  under the canonical mapping  $T(V) \rightarrow C(Q)$  then we get an  $N$ -filtering on  $C(Q)$  with  $C_0 = K$  and with  $C_h C_k \subset C_{h+k}$ . Let  $G = \bigoplus_{h \geq 0} G^h$  with  $G^h = C_h / C_{h-1}$  if  $h > 0$  and with  $G^0 = K$ .  $G$  is an

algebra with the following multiplication: for  $\xi \in G^h$  and  $\eta \in G^k$  we have  $\xi\eta = xy + C_{h+k-1}$  if  $\xi = x + C_{h-1}$  and  $\eta = y + C_{k-1}$ . One can prove that  $G$  and  $\Lambda(V)$  are isomorphic algebras so that we can identify them. It follows that  $\xi\eta - (-1)^{hk}\eta\xi = 0$ . In analogy with the Poisson bracket on  $S(\mathfrak{g})$  we define

$\{\xi, \eta\}_{\epsilon} = xy - (-1)^{hk}xy + C_{h+k-3}$ . It is not difficult to prove that  $\{\alpha, \xi\eta\}_{\epsilon} = \{\alpha, \xi\}_{\epsilon}\eta + (-1)^{jh}\xi\{\alpha, \eta\}_{\epsilon}$  for  $\alpha = a + C_{j-1}$ , that

$\{\xi, \eta\}_\epsilon + (-1)^{hk} \{\eta, \xi\}_\epsilon = 0$  and that  
 $\{\xi, \{\eta, \zeta\}_\epsilon\}_\epsilon = \{\{\xi, \eta\}_\epsilon, \zeta\}_\epsilon + (-1)^{hk} \{\eta, \{\xi, \zeta\}_\epsilon\}_\epsilon$  ("Jacobi").  
 If we transport  $\{, \}_\epsilon$  to  $\Lambda(V)$  and if  $x$  and  $y \in V$  then  $\{x, y\}_\epsilon = \Phi(x, y)$ . In  
 the notation of BOURBAKI:  $\{x, u\}_\epsilon = i_x^\Phi(u)$  with  $x \in V$  and  $u \in \Lambda(V)$ .

Let  $x$  be the product of  $h$  elements of  $V$  and  $y$  be the product of  $k$  elements  
 of  $V$  then we define in  $C(Q)$  and in  $\Lambda(V)$  the  $\epsilon$ -commutator of  $x$  and  $y$   
 by the relation  $[x, y]_\epsilon = xy - (-1)^{hk}yx$  resp.  $= x \wedge y - (-1)^{hk}y \wedge x$ . It can  
 be proved that  $\eta(\{x, y\}_\epsilon) = [\eta(x), \eta(y)]_\epsilon$  if one of the elements  $x$  and  $y$  lies  
 in  $V$ .

**REMARK.** The formula  $\eta(\{x, y\}_\epsilon) = [\eta(x), \eta(y)]_\epsilon$  and the analogous one  
 $\eta(\{x, y\}) = [\eta(x), \eta(y)]$  in the case of Lie algebras are in general *not true* if  
*both*  $x$  and  $y$  are not in  $V$ .

Let  $V$  be the space with base  $(q_1, \dots, q_n, p_1, \dots, p_n)$  again. Let now  $\Phi$  be  
 a *symmetric* bilinear form on  $V$  with  $\Phi(p_i, p_j) = \Phi(q_i, q_j) = 0$  ( $\forall i, j$ ) and  
 with  $\Phi(q_i, p_j) = \delta_{ij}$ .  $\Phi$  is non degenerate and  $Q$  with  $Q(x) = \frac{1}{2}\Phi(x, x)$  is  
 a quadratic form with  $\Phi$  as bilinear form. We get in this way a Clifford  
 algebra  $C(Q)$  with  $[p_i, p_j]_\epsilon = [q_i, q_j]_\epsilon = 0$  and with  $[q_i, p_j]_\epsilon = \delta_{ij}$ . The Pois-  
 son bracket on  $\Lambda(V)$  is given by  $\{p_i, p_j\}_\epsilon = \{q_i, q_j\}_\epsilon = 0$  and  $\{q_i, p_j\}_\epsilon = \delta_{ij}$ .

**REMARK.** We have seen quite a lot of similarities between the symmetric  
 and enveloping algebra of a Lie algebra on the one hand and the exterior  
 and Clifford algebra on the other hand. There is another similarity: the  
 envelope contains a Lie algebra; the Clifford algebra contains (the spin  
 representation of) a orthogonal (with respect to  $Q$ ) group.

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# Classical and quantum representation theory

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These notes present an introduction to an analytic version of deformation quantization. The central point is to study algebras of physical observables and their irreducible representations. In classical mechanics one deals with real Poisson algebras, whereas in quantum mechanics the observables have the structure of a real non-associative Jordan-Lie algebra. The non-associativity is proportional to  $\hbar^2$ , hence for  $\hbar \rightarrow 0$  one recovers a real Poisson algebra. This observation lies at the basis of 'strict' deformation quantization, where one deforms a given Poisson algebra into a  $C^*$ -algebra, in such a way that the basic algebraic structures are preserved.

Our main interest lies in degenerate Poisson algebras and their quantization by non-simple Jordan-Lie algebras. The traditional symplectic manifolds of classical mechanics, and their quantum counterparts (Hilbert spaces and operator algebras which act irreducibly) emerge from a generalized representation theory. This two-step procedure sheds considerable light on the subject.

We discuss a large class of examples, in which the Poisson algebra canonically associated to an (integrable) Lie algebroid is deformed into the Jordan-Lie algebra of the corresponding Lie groupoid. A special case of this construction, which involves the gauge groupoid of a principal fibre bundle, describes the classical and quantum mechanics of a particle moving in an external gravitational and Yang-Mills field.

## 1. INTRODUCTION

In quantization theory one tries to establish a correspondence between a classical mechanical system, and a quantum one. The traditional method, already contained in the work of Heisenberg and Dirac, is canonical quantization. Attempts to generalize this procedure, and put it on a solid mathematical footing have led to geometric quantization [49, 24, 20]. This is a certain algorithm which still contains many gaps, and for various reasons cannot be considered satisfactory [44]. The same comment applies to path integral quantization, but

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we hasten to remark that both techniques have led to many examples, constructions, and insights, in physics as well as mathematics, that would have been hard to reach otherwise, and still provide the main testing ground for alternative methods.

One such alternative method is deformation quantization. The version that we use (and partly propose) employs techniques from algebra, differential geometry, and functional analysis, and appears to be very interesting from a mathematical point of view. One attempts to relate Poisson algebras to  $C^*$ -algebras in a way specified below, and as such it is possible to relate to, and exploit the phenomenal progress made in both subjects over the last decade. This progress has consisted of discovering and understanding general structures through specific examples, and in a certain sense a unification of the three mathematical disciplines mentioned above has been achieved, under the name of non-commutative geometry. On the operator-algebraic side this includes cyclic cohomology of operator algebras [14] and operator K-theory (non-commutative topology) [42], which have found interesting applications (highly relevant to quantization theory!) in foliation theory and generalized index theorems [32]. As to Poisson algebras, we mention Poisson cohomology [23] and the theory of symplectic groupoids [48].

From the point of view of physics we wish to stress that the quantization procedure discussed here is very satisfactory in that it places physical notions like observables and states at the forefront (inspired by algebraic quantum field theory [21]), plays down the (quite unnecessary) use of complex numbers in quantum mechanics, and accurately describes a large class of examples relevant to Nature. Moreover, it brings classical and quantum mechanics very closely together and highlights their common structures.

We will introduce the relevant mathematical structures step by step, on the basis of the familiar Weyl quantization of a particle moving on  $\mathbb{R}^n$ . This will lead us to Poisson algebras, Jordan-Lie algebras, and  $C^*$ -algebras. We then introduce the appropriate notion of a representation of each of these objects, and motivate an irreducibility condition. Lie groups form a rich class of examples on which to illustrate the general theory, but since these only describe particles with nothing but an internal degree of freedom, we must look elsewhere for structures describing genuine physics. A rich structure that is tractable by our methods, and at the same time describes real physical systems, is that of a Lie groupoid [31, 16]. It has an associated ‘infinitesimal’ object (a Lie algebroid), and, as we will explain, the passage from an algebroid to a groupoid essentially amounts to quantization.

## 2. CLASSICAL MECHANICS AND POISSON ALGEBRAS

### 2.1. *Introductory example: particle on flat space*

Consider a particle moving on the configuration space  $Q = \mathbb{R}^n$ . We use canonical co-ordinates  $(x^\mu, p_\mu)$  (usually simply written as  $(x, p)$ ) on the cotangent bundle  $M = T^*\mathbb{R}^n$  ( $\mu = 1, \dots, n$ ), so that  $(x, p)$  stands for the one-form

$p_\mu dx^\mu \in T_x^* \mathbb{R}^n$ . In mechanics a key role is played by the Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu} \frac{\partial g}{\partial x^\mu}, \quad (2.1)$$

where  $f_1, f_2 \in C^\infty(M)$ . Here  $C^\infty(M) \equiv A_0$  stands for the real vector space of real-valued smooth functions on  $M$ . Its elements are classical observables. Apart from the Poisson bracket, there is another bilinear map from  $A_0 \otimes_{\mathbb{R}} A_0 \rightarrow A_0$ , namely the ordinary (pointwise) multiplication  $\cdot$ . Let us write  $f\sigma g$  for  $fg$  ( $\equiv f \cdot g$ ), and  $f\alpha g$  for  $\{f, g\}$ . The algebraic operations  $\sigma$  and  $\alpha$  satisfy the following properties:

1.  $f\sigma g = g\sigma f$  (symmetry);
2.  $f\alpha g = -g\alpha f$  (anti-symmetry);
3.  $(f\alpha g)\alpha h + (h\alpha f)\alpha g + (g\alpha h)\alpha f = 0$  (Jacobi identity);
4.  $(f\sigma g)\alpha h = f\sigma(g\alpha h) + g\sigma(f\alpha h)$  (Leibniz rule);
5.  $(f\sigma g)\sigma h = f\sigma(g\sigma h)$  (associativity).

The meaning of  $\alpha$  and  $\sigma$  is as follows. To start with the latter, we remark that the spectrum  $\text{spec}(f)$  of a function  $f \in C^\infty(M)$  is the set of values it takes (that is, the possible values that the observable  $f$  may have). If  $f$  is concretely given (i.e., we know " $f(m_1) = a_1, f(m_2) = a_2 \dots$ ") then we obviously know the spectrum immediately. However,  $f$  may be regarded as an abstract element of the algebra  $A_0$ . The point is now that  $\text{spec}(f)$  is completely determined by its location in  $A_0$ , equipped with the product  $\sigma$  (forgetting the Poisson bracket). Namely, if  $a \in \text{spec}(f)$  then  $f - a1$  (where 1 is the function on  $M$  which is identically equal to 1) fails to have an inverse in  $A_0$ , whereas, conversely,  $(f - a1)^{-1}$  is a well-defined element of  $A_0$ , satisfying  $(f - a1)^{-1}\sigma(f - a1) = 1$  if  $a \notin \text{spec}(f)$ . Hence we may *define*  $\text{spec}(f)$  as the set of real numbers  $a$  for which  $f - a1$  fails to have an inverse in  $A_0$ . A closely related point is that  $\sigma$  allows one to define functions of observables (starting from  $f^2 = f\sigma f$ ); this is related to the previous point via the spectral calculus.

The Poisson bracket  $\alpha$  determines the role any observable plays as the generator of a flow on the space  $M$  of pure states on  $A_0$ . To explain this, we need to introduce the concept of the state space of an algebra. The state space  $S(A)$  of a real algebra  $A$  may be defined as the space of normalized positive functionals on  $A$ , i.e., the linear maps  $\omega : A \rightarrow \mathbb{R}$  which satisfy  $\omega(f^2) \geq 0$  for all  $f \in A$ , and  $\omega(1) = 1$ . If  $\omega_1$  and  $\omega_2$  are states then  $\lambda\omega_1 + (1 - \lambda)\omega_2$  is a state if  $\lambda \in [0, 1]$ . A state is defined to be pure if it does not allow such a decomposition unless  $\lambda = 0, 1$ ; otherwise, it is called mixed. The physical interpretation of  $\omega(f)$  is that this number equals the expectation value of the observable  $f$  in the state  $\omega$ . Any point  $m$  of  $M$  defines a pure state on  $A_0$  by  $m(f) = f(m)$ , and these in fact exhaust the set of pure states. This statement holds equally well if we had taken  $A_0$  to be  $C_c^\infty(M)$  or  $C_0^\infty(M)$  (the smooth

functions with compact support, and those vanishing at infinity, respectively), but the pure state space of  $C_b^\infty(M)$  (the bounded smooth functions) is the so-called Cech-Stone compactification of  $M$ .

Back to the Poisson bracket, each  $f \in A_0$  defines a so-called Hamiltonian vector field  $X_f$  on  $M$  by

$$X_f g = \{g, f\}, \quad (2.2)$$

and this generates a Hamiltonian flow  $\phi_t^f$  on  $M$  (as the solution of the differential equation  $d\phi_t^f/dt = X_f(\phi_t^f)$ ), cf. [1, 29]. That  $X_f$  is indeed a vector field (i.e., a derivation of  $C^\infty(M)$ ) is a consequence of the Jacobi identity on  $\alpha$ .

The example  $M = T^*\mathbb{R}^n$  has the following feature: any two points of  $M$  can be connected by a (piecewisely) smooth Hamiltonian flow. This property is equivalent to the following:  $\{X_f(m) | f \in A_0\} = T_m M$  for all  $m \in M$ . That is, the Hamiltonian vector fields span the tangent space at any point of  $M$ .

To sum up, observables take values, and one may define functions of them, which two properties are determined by the product  $\sigma$ ; moreover, they generate flows of the pure state space, which are determined by the Poisson bracket  $\alpha$ .

## 2.2. Poisson algebras and their representations

**DEFINITION 1.** *A Poisson algebra is a vector space  $A$  over the real numbers, equipped with two bilinear maps  $\alpha, \sigma : A \otimes_{\mathbb{R}} A \rightarrow A$  which satisfy the five conditions in the preceding subsection.*

The examples of Poisson algebras we will consider are of the type  $A = C^\infty(M)$  for some manifold  $M$ , which has a Poisson structure, in the sense that  $\alpha$  is some Poisson bracket and  $\sigma$  is multiplication. In that case,  $M$  together with the Poisson structure is called a Poisson manifold. If  $M$  has the special feature discussed after (2.2) that any two points can be joined by a piecewisely smooth Hamiltonian curve, then  $M$  is called symplectic. If not, we can impose an equivalence relation [47]  $\sim$  on  $M$ , under which  $x \sim y$  iff  $x$  and  $y$  can be joined by a piecewisely smooth Hamiltonian curve. The equivalence class  $L_x$  of any point can be shown to be a manifold, which is embedded in  $M$ . If  $i$  is the embedding map then the relation  $\{i^*f, i^*g\}_{L_x} = i^*\{f, g\}_M$  defines a Poisson structure  $\{, \}_{L_x}$  on  $L_x$ , which is obviously symplectic, and we call  $L_x$  a symplectic leaf of  $M$ . More advanced considerations show that any Poisson manifold is foliated by its symplectic leaves [29].

If  $M = S$  is symplectic then the Poisson bracket can be derived from a symplectic form on  $S$  [1, 29]. The corresponding  $A = C^\infty(S)$  are in some sense the ‘canonical models’ of Poisson algebras. This motivates the following

**DEFINITION 2.** *A representation of a Poisson algebra  $A$  is a map  $\pi_c^S : A \rightarrow C^\infty(S)$ , where  $S$  is a symplectic manifold, satisfying the following conditions for all  $f, g \in A$ :*

1.  $\pi_c^S(\lambda f + \mu g) = \lambda \pi_c^S(f) + \mu \pi_c^S(g)$ , for all  $\lambda, \mu \in \mathbb{R}$  (linearity);



2.  $\pi_c^S(fg) = \pi_c^S(f)\pi_c^S(g)$  (preserves  $\sigma$ );
3.  $\pi_c^S(\{f, g\}_M) = \{\pi_c^S(f), \pi_c^S(g)\}_S$  (preserves  $\alpha$ );
4. The vector field  $X_{\pi_c^S(f)}$  is complete if  $X_f$  is (self-adjointness).

The  $c$  in  $\pi_c^S$  stands for ‘classical’, and the above defines a ‘classical’ representation (as opposed to a ‘quantum’ representation of algebraic objects by operators on a Hilbert space; as we shall see later, the distinction between classical and quantum is actually blurred). A vector field is called complete if its flow exists for all times. If  $f$  had compact support then its flow is automatically complete [1]. Condition 4 excludes situations of the following type. Take  $M = T^*\mathbb{R}$  with the usual Poisson structure (2.1), and take  $S$  any open set in  $M$ . If  $i$  is the embedding of  $S$  into  $M$ , with the Poisson structure borrowed from  $M$  by restriction, then  $\pi_c^S(f) = i^*f$  satisfies 1-3 but not 4 (unless  $S = M$ ).

The following theorem shows that all representations are actually of the type  $\pi_c^S = J^*$ , where  $J : S \rightarrow M$  is a Poisson morphism.

**THEOREM 1.** *Let  $M$  be a finite-dimensional Poisson manifold,  $A = C^\infty(M)$  the corresponding Poisson algebra, and let  $\pi_c^S : A \rightarrow C^\infty(S)$  be a representation of  $A$ . Then there exists a map  $J : S \rightarrow M$  such that  $\pi_c^S = J^*$ .*

*Proof.* For the elementary  $C^*$ -algebra theory used in the proof, cf. e.g. [33, 43, 9]. Take  $s \in S$ , and define a linear functional  $\tilde{J}(s)$  on  $C_0^\infty(M)$  by putting  $\langle \tilde{J}(s), f \rangle = (\pi_c^S(f))(s)$  for  $f \in C_0^\infty(M)$ . By property 2 of a representation,  $\tilde{J}(s)$  is multiplicative, hence positive and continuous, so it extends to a pure state on the commutative  $C^*$ -algebra  $C_0(M)$  (which is the complexification of the norm-closure of  $C_0^\infty(M)$ ). Hence by the Gel’fand isomorphism  $\tilde{J}(s)$  corresponds to a point  $J(s)$  of  $M$ . Hence we have found the required map  $J : S \rightarrow M$ .  $\square$

For reasons to emerge in subsect. 2.3 below, we will refer to  $J$  as the *generalized moment map*. Property 3 of a representation implies that  $J$  is what is called a Poisson morphism. Such maps have been studied extensively in the literature [47, 29]. The self-adjointness condition 4 translates into a condition on  $J$ , which is called *completeness* by A. Weinstein. Examples suggest that it is actually a classical analogue of the condition on representations of real operator algebras on Hilbert spaces that these preserve self-adjointness (a special case of which is the familiar requirement that group representations be unitary). However, this self-adjointness condition is actually a completeness condition, too, for it guarantees that the unitary flow on Hilbert space generated by the self-adjoint representative of a given operator can be defined for all times (also cf. sect. 3 below). Further conditions on  $\pi_c^S$  could be imposed to guarantee that  $J$  is smooth, but as far as we can see we can develop the theory without those.

The following proposition (which is well known, cf. [47, 29]) is crucial for the analysis of irreducible representations (to be defined shortly). Here  $J_*$  denotes the push-forward of  $J$  [1].

PROPOSITION 1. Let  $J : S \rightarrow M$  be the Poisson morphism corresponding to a representation  $\pi_c^S$  of the Poisson algebra  $C^\infty(M)$ . Then for any  $f \in C^\infty(M)$

$$J_* X_{\pi_c^S(f)} = X_f, \quad (2.3)$$

where  $X_f$  is the Hamiltonian vector field defined by  $f$  (etc.). Moreover, the image of the flow of  $X_{\pi_c^S(f)}$  under  $J$  is the flow of  $X_f$ .

*Proof.* Take  $g \in C^\infty(M)$  arbitrary. By definition of  $\pi_c^S$  and  $J$ , we have

$$\{\pi_c^S(f), \pi_c^S(g)\}_S(s) = \{f, g\}_M(J(s)) \quad (2.4)$$

Upon use of (2.2), this leads to the identity  $(J_* X_{\pi_c^S(f)} g)(J(s)) = (X_f g)(J(s))$ , whence the result.  $\square$

Since  $S$  is symplectic, the symplectic form  $\omega$  provides an isomorphism  $\tilde{\omega} : T_s^* S \rightarrow T_s S$  for any  $s \in S$ . This is given by  $\tilde{\omega}(df) = X_f$ , or  $\tilde{\omega}^{-1}(X) = i_X \omega$  (evaluated at any point  $s$ ). Now let  $\tilde{T}_s S$  denote the subspace of  $T_s S$  which is spanned by Hamiltonian vectors (i.e., of the form  $X_{\pi_c^S(f)}$ ,  $f \in C^\infty(M)$ , taken at  $s$ ). Then

$$\tilde{T}_s S = \tilde{\omega} \circ J^*(T_{J(s)}^* M), \quad (2.5)$$

and  $\tilde{\omega}$  is a bijection between  $\tilde{T}_s S$  and  $J^*(T_{J(s)}^* M)$ , where  $J^*$  is the pull-back of  $J$  (to 1-forms, in this case). This follows rapidly from the preceding proposition.

DEFINITION 3. A representation  $\pi_c^S$  of a Poisson algebra  $C^\infty(M)$  is called *irreducible* if

$$\{X_{\pi_c^S(f)}(s) | f \in C^\infty(M)\} = T_s S \quad \forall s \in S. \quad (2.6)$$

As mentioned before in a different variant, this condition guarantees that any two points in  $S$  can be joined by a piecewise smooth curve, whose tangent vector field is of the form  $X_{\pi_c^S(f)}$ . Of course, since  $S$  is symplectic any two points can be joined by such a curve with tangent vectors  $X_g$ ,  $g \in C^\infty(S)$ , even if  $\pi_c^S$  is not irreducible, but one may not be able to take  $g = \pi_c^S(f)$ . Note, that we could have broadened our definition of a representation by allowing  $S$  to be a Poisson manifold; in that case, however, the irreducibility condition would force  $S$  to be symplectic anyway. In the literature [47, 29] people appear to be mainly interested in the opposite situation, where a Poisson morphism  $J : S \rightarrow M$  ( $S$  symplectic) is called *full* if (in our language) the corresponding representation  $\pi_c^S = J^*$  is faithful. As the following result shows, this is indeed quite opposite to an irreducible representation, which has a large kernel unless  $M$  is symplectic itself.

THEOREM 2. If a representation  $\pi_c^S : C^\infty(M) \rightarrow C^\infty(S)$  of a Poisson algebra is irreducible then  $S$  is symplectomorphic to a covering space of a symplectic leaf of  $M$ .

*Proof.* We first show that  $J : S \rightarrow M$  is an immersion. Namely, if  $J_*X = 0$  for some  $X \in T_s S$  then  $\langle J_*X, \theta \rangle_{J(s)} = \langle X, J^*\theta \rangle_s = 0$ , but by (2.5) and (2.6) any  $\theta' \in T^*S$  may be written as  $\theta' = J^*\theta$  for some  $\theta \in T_{J(s)}^*M$ . Hence  $X = 0$ , and  $J$  is an immersion. Since  $J$  is a Poisson morphism, it follows that  $S$  is locally symplectomorphic to  $J(S) \subset M$ .

Next,  $J(S)$  must actually be a symplectic leaf of  $M$ . For suppose that there is a proper inclusion  $J(S) \subset L$ , where  $L$  is a symplectic leaf of  $M$ . It follows from the Darboux-Weinstein theorem [1, 29] that any point  $x$  in a symplectic space has a neighbourhood  $U_x$  such that any two points in  $U$  may be connected by a smooth Hamiltonian curve. If we take  $x$  to lie on the boundary of  $J(S)$  in  $L$ , then we find that there exist  $m_1 \in J(S)$  and  $J(S) \not\ni m_2 \in L$  which can be connected by a smooth curve  $\gamma$  with tangent vector field  $X_f$ , for some  $f \in C^\infty(M)$ . Let  $m_1 = J(s_1)$ , and consider the flow  $\tilde{\gamma}$  of  $X_{\pi_c^S(f)}$  starting at  $s_1$ . By the proposition above,  $J \circ \tilde{\gamma} = \gamma$ . However, since  $m_2 \notin J(S)$ , the flow  $\tilde{\gamma}$  must suddenly stop, which contradicts the self-adjointness (completeness) property 4 of a representation. Hence to avoid a contradiction we must have  $J(S) = L$ .

A similar argument shows that  $J : S \rightarrow J(S)$  must be a covering projection. For  $J$  not to be a covering projection, there must exist a point  $m \in M$ , a neighbourhood  $V_m$  of  $m$ , and a connected component  $J_i^{-1}(V_m)$  of  $J^{-1}(V_m)$ , so that  $J(J_i^{-1}(V_m)) \subset V_m$  is a proper inclusion. But in that case we could choose points  $s_1 \in J(J_i^{-1}(V_m))$  and  $J(J_i^{-1}(V_m)) \not\ni s_2 \in V_m$  which can be connected by a smooth hamiltonian curve, and arrive at a contradiction to the self-adjointness property of  $\pi_c^S$ .  $\square$

### 2.3. The Lie-Kirillov-Kostant-Souriau Poisson structure

We obtain a basis class of Poisson algebras by taking  $M = \mathfrak{g}^*$ , which is the dual of the Lie algebra  $\mathfrak{g}$  of some Lie group  $G$ . We may regard  $X \in \mathfrak{g}$  as an element of  $C^\infty(\mathfrak{g}^*)$ , by  $X(\theta) = \langle \theta, X \rangle$ , and the Poisson structure of  $\mathfrak{g}^*$  is completely determined by putting

$$\{X, Y\} = [X, Y] \quad (2.7)$$

(cf. [1, 29] for more information). The classical algebra of observables  $C^\infty(\mathfrak{g}^*)$  describes a particle which doesn't move, but only has an internal degree of freedom (e.g., spin if  $G = SU(2)$ ).

Let  $\pi_c^S : C^\infty(\mathfrak{g}^*) \rightarrow C^\infty(S)$  be a representation of  $C^\infty(M)$ , with  $S$  connected. For each  $X \in \mathfrak{g}$  we define a function  $f_X$  on  $S$  by

$$f_X = \pi_c^S(X). \quad (2.8)$$

By definition of a representation

$$\{f_X, f_Y\}_S = f_{[X, Y]}. \quad (2.9)$$

If  $\tilde{X}$  is the Hamiltonian vector field defined by  $f_X$  (so that  $\tilde{X}g = \{g, f_X\}_S$ ) then (2.9) and the Jacobi identity imply that  $[\tilde{X}, \tilde{Y}] = -[\tilde{X}, \tilde{Y}]$  (where the first

bracket is the commutator of vector fields and the second one is the Lie bracket on  $\mathfrak{g}$ ). By self-adjointness, the flow  $\varphi_t^X$  of  $\tilde{X}$  is defined for all  $t$ , and this leads to an action  $\pi_c^S$  of  $\exp X \in G$  on  $S$  by  $\pi_c^S(\exp X)s = \varphi_1^X(s)$ . If  $G$  is simply connected this eventually defines a proper symplectic action of  $G$  on  $S$ .

Conversely, let  $G$  act on a symplectic manifold  $S$  so as to preserve the symplectic form  $\omega$ . We may then define a vector field  $\tilde{X}$  for each  $X \in \mathfrak{g}$  by

$$(\tilde{X}f)(s) = \frac{d}{dt}f(e^{tX}s)|_{t=0}, \quad (2.10)$$

where we have written the action of  $x \in G$  on  $s \in S$  simply as  $xs$ . The action is called Hamiltonian  $i_{\tilde{X}}\omega = df_X$  for some  $f_X \in C^\infty(S)$  (this is guaranteed if  $H^1(S, \mathbb{R}) = 0$ ), and strongly Hamiltonian if (2.9) is satisfied on top of that. If the former condition is met, one can define a map  $J : S \rightarrow \mathfrak{g}^*$  by means of

$$\langle J(s), X \rangle = f_X(s), \quad (2.11)$$

with pull-back  $J^* : C^\infty(\mathfrak{g}^*) \rightarrow C^\infty(S)$ . In that case we clearly see from (2.8) that the map  $J$  defined by (2.11) is a special case of the generalized moment map constructed in Theorem 1. Indeed,  $J$  in (2.11) is called the moment(um) map in the literature [20, 1, 29]. (Note the varying sign conventions. We follow [1] in putting  $i_{\tilde{X}}\omega = df_X$ ,  $X_f g = \{g, f\}$ , and  $\{f_X, f_Y\}_S = f_{[X, Y]}$ , but the alternative convention  $i_{\tilde{X}}\omega = -df_X$ ,  $X_f g = \{f, g\}$ , and  $\{f_X, f_Y\}_S = -f_{[X, Y]}$  occurs as well.)

If the symplectic  $G$ -action on  $S$  is Hamiltonian but not strongly so, the right-hand side of (2.9) acquires an extra term, and this situation may be analyzed in terms of Lie algebra cohomology [20, 1, 29]. The result is that the Poisson bracket (2.7) can be modified, so that  $\pi_c^S = J^*$  defines a representation of  $C^\infty(\mathfrak{g}^*)$ , equipped with the modified Poisson structure.

In the strongly Hamiltonian case  $J^*$  produces a representation  $\pi_c^S \equiv J^*$  of  $C^\infty(\mathfrak{g}^*)$  equipped with the Lie-Kirillov-Kostant-Souriau Poisson structure (2.7). The fact that  $J$  is a Poisson morphism may be found in [1, 29, 20], and it remains to check the self-adjointness condition. We observe that vector fields on  $S$  of the type  $X_{\pi_c^S(f)}$  ( $f \in C^\infty(\mathfrak{g}^*)$ ) are tangent to a  $G$ -orbit, so that their flow  $\gamma_t^{\pi_c^S(f)}$  cannot map a point of  $S$  into a different orbit. This reduces the situation to the case where  $G$  acts transitively on  $S$ . In that case, the vector fields  $\{\tilde{X}|X \in \mathfrak{g}\}$  span the tangent space of  $S$  at any point, so that  $\pi_c^S$  is irreducible. By Theorem 2, the image of  $J$  must be a symplectic leaf of  $\mathfrak{g}^*$ , hence a co-adjoint orbit (this shows, incidentally, that the famous Kostant-Souriau theorem which asserts that any symplectic space which allows a transitive strongly Hamiltonian action of a Lie group  $G$  is symplectomorphic to a covering space of a co-adjoint orbit of  $G$  [20, 29] is a special case of our Theorem 2). Now take  $f \in C^\infty(\mathfrak{g}^*)$  with Hamiltonian vector field  $X_f$  and flow  $\gamma_t^f$ . Since (by definition)  $G$  acts transitively on any co-adjoint orbit in  $\mathfrak{g}^*$ , we may write  $\gamma_t^f(\theta) = \pi_{\text{co}}(x_t)\theta$  for some curve  $x_t$  in  $G$  (not uniquely defined, and dependent on the argument  $\theta \in \mathfrak{g}^*$ ); here  $\pi_{\text{co}}$  is the co-adjoint

representation of  $G$  on  $\mathfrak{g}^*$ . We now use Proposition 1 and the equivariance of  $J$  (that is,  $J \circ x = \pi_{\text{co}}(x) \circ J$  [20, 1, 29]) to derive  $J \circ x_t(s) = J \circ \gamma_t^{\pi_c^S(f)}(s)$  for any  $s \in S$ ; here  $x_t$  depends on  $J(s)$ . Since  $J$  is an immersion this implies  $\gamma_t^{\pi_c^S(f)}(s) = x_t(s)$ , hence  $\gamma_t^{\pi_c^S(f)}$  is defined whenever  $x_t$  is; in particular, if  $\gamma_t^f$  is complete then  $\gamma_t^{\pi_c^S(f)}$  is, so that the representation  $\pi_c^S$  is self-adjoint.

In passing, we have observed that the irreducible representations of  $C^\infty(\mathfrak{g}^*)$  are given by the co-adjoint orbits in  $\mathfrak{g}^*$  (and their covering spaces).

### 3. QUANTUM MECHANICS AND JORDAN-LIE ALGEBRAS

#### 3.1. Weyl quantization on flat space

To introduce some relevant mathematical structures in a familiar context we briefly review the Weyl quantization procedure of a particle moving on  $Q = \mathbb{R}^n$ , with phase space  $M = T^*\mathbb{R}^n$  (as in subsect. 2.1). It is convenient to introduce a partial Fourier transform of  $f \in C^\infty(M)$  by

$$\check{f}(x, \dot{x}) = \int \frac{d^n p}{(2\pi)^n} e^{ip\dot{x}} f(x, p); \quad (3.1)$$

this makes  $\check{f}$  a function on the tangent bundle  $T\mathbb{R}^n$ , where we use canonical co-ordinates  $(x, \dot{x}) \equiv \dot{x}^\mu \partial / \partial x^\mu \in T_x \mathbb{R}^n$ . For (3.1) to make sense we must have that  $f$  is integrable in the fiber direction (i.e., over  $p$ ). Let  $f$  be such that  $\check{f} \in C_c^\infty(T\mathbb{R}^n)$ ; we refer to this class of functions as  $\overline{\mathfrak{A}}_0$ . We then define an operator  $Q_\hbar(f)$  on the Hilbert space  $\mathcal{H} = L^2(\mathbb{R}^n)$  by

$$(Q_\hbar(f)\psi)(x) = \int d^n y \tilde{Q}_\hbar(f)(x, y)\psi(y), \quad (3.2)$$

with kernel

$$\tilde{Q}_\hbar(f)(x, y) = \hbar^{-n} \check{f}\left(\frac{x+y}{2}, \frac{x-y}{\hbar}\right). \quad (3.3)$$

This operator is compact (it is even Hilbert-Schmidt, since the kernel is in  $C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ ), and thus it is bounded. (The norm of an operator  $T$  on a Hilbert space  $\mathcal{H}$  is defined by  $\|T\| = \sup_\psi (T\psi, T\psi)^{1/2}$ , where the supremum is over all vectors  $\psi$  of unit length. An operator  $T$  is called bounded if this norm is finite. An operator is called compact if it may be approximated in norm by operators with a finite-dimensional range [36]. Compact operators behave to some extent like finite-dimensional matrices).

A crucial property of  $Q_\hbar(f)$  is that it is self-adjoint (since  $f$  is real-valued). This means, that  $Q_\hbar$  may be regarded as a map from  $\overline{\mathfrak{A}}_0$  into  $\mathfrak{A} = \mathcal{K}(\mathcal{L}^2(\mathbb{R}^n))_{\text{sa}}$  (the set of self-adjoint compact operators on  $\mathcal{H} = L^2$ ). As a real subspace of  $\mathcal{B}(\mathcal{H})$ ,  $\mathfrak{A}$  is itself a normed space, which is, in fact, complete (because  $\mathcal{K}(\mathcal{H})$  is). We can make  $\overline{\mathfrak{A}}_0$  into a real Banach space, too, by equipping it with the norm

$$\|f\|_0 = \sup_{m \in M} |f(m)|. \quad (3.4)$$

The completion of  $\overline{\mathfrak{A}_0}$  under this norm is  $\mathfrak{A}_0 = \mathfrak{C}_0(\mathfrak{M}, \mathbb{R})$  (the space of real-valued continuous functions on  $M$  which vanish at infinity).

We interpret  $Q_{\hbar}(f)$  as the quantum observable corresponding to the classical observable  $f$ . Accordingly, we call  $\mathfrak{A}$  the (quantum) algebra of observables (of a particle on  $\mathbb{R}^n$ ). As in the classical case, we may identify two algebraic operations on  $\mathfrak{A}$  (that is, bilinear maps  $\mathfrak{A} \otimes_{\mathbb{R}} \mathfrak{A} \rightarrow \mathfrak{A}$ ). They are

$$A\sigma_{\hbar}B = \frac{1}{2}(AB + BA); \quad A\alpha_{\hbar}B = \frac{1}{i\hbar}(AB - BA). \quad (3.5)$$

The latter depends on  $\hbar$ , so we will rename  $\mathfrak{A}$ , equipped with  $\sigma_{\hbar}$  and  $\alpha_{\hbar}$ , as  $\mathcal{A}_{\hbar}$  (the norm  $\|\cdot\|$  does not depend on  $\hbar$ ). One may verify the following properties:

1.  $A\sigma_{\hbar}B = B\sigma_{\hbar}A$  (symmetry);
2.  $A\alpha_{\hbar}B = -B\alpha_{\hbar}A$  (anti-symmetry);
3.  $(A\alpha_{\hbar}B)\alpha_{\hbar}C + (C\alpha_{\hbar}A)\alpha_{\hbar}B + (B\alpha_{\hbar}C)\alpha_{\hbar}A = 0$  (Jacobi identity);
4.  $(A\sigma_{\hbar}B)\alpha_{\hbar}C = A\sigma_{\hbar}(B\alpha_{\hbar}C) + B\sigma_{\hbar}(A\alpha_{\hbar}C)$  (Leibniz rule);
5.  $(A\sigma_{\hbar}B)\sigma_{\hbar}C - A\sigma_{\hbar}(B\sigma_{\hbar}C) = \frac{\hbar^2}{4}(A\alpha_{\hbar}C)\alpha_{\hbar}B$  (weak associativity);
6.  $\|A\sigma_{\hbar}B\| \leq \|A\| \|B\|$  (submultiplicativity of the norm);
7.  $\|A^2\| \leq \|A\|^2$  (spectral property of the norm).

We see that 1-4 are identical to the corresponding properties of a Poisson algebra, and 5 implies that we are now dealing with a deformation of the latter in a non-associative direction, in that the symmetric product  $\sigma_{\hbar}$  is now non-associative. A weak form of associativity does hold, this is the so-called associator identity

$$(A^2\sigma_{\hbar}B)\sigma_{\hbar}A = A^2\sigma_{\hbar}(B\sigma_{\hbar}A), \quad (3.6)$$

which can be derived from 1-5. The last two properties imply  $\|A^2\| = \|A\|^2$  [45], which leads to the usual spectral calculus.

Before commenting on the general structure we have found, let us find the meaning of the products  $\sigma_{\hbar}$  and  $\alpha_{\hbar}$  (cf. subsect. 2.1). We start with  $\sigma_{\hbar}$ . In classical and quantum mechanics alike, the spectrum of a self-adjoint operator is identified with the values the corresponding observable may assume. We have seen that the spectrum of a classical observable is determined by the symmetric product  $\sigma$ . In standard Hilbert space theory (which is applicable, as we have realized  $\mathfrak{A}$  as a set of operators acting on  $\mathcal{H} = L^2$ ) the spectrum of a self-adjoint operator  $A$  is defined as the set of values of  $z$  for which the resolvent  $(A - z)^{-1}$  fails to exist as an element of  $\mathcal{B}(\mathcal{H})$  [36]. More abstractly, the spectrum of an element  $A$  of a  $C^*$ -algebra  $\mathcal{B}$  is defined by replacing  $\mathcal{B}(\mathcal{H})$  by  $\mathcal{B}$  in the above [43, 9]. In fact, this definition only uses the anti-commutator (rather than the associative operator product, which combines the anti-commutator

and the commutator), so that we conclude that the symmetric product on the algebra of observables determines the spectral content. This observation is originally due to Segal [40] (and was undoubtedly known to von Neumann, who introduced the anti-commutator), and a quick way to see this is that the spectrum of  $A$  is determined by the  $C^*$ -algebra  $C^*(A)$  it generates; this is a *commutative* sub-algebra of  $\mathcal{B}$  (or,  $\mathcal{K}(\mathcal{H})$  in our example above) which clearly only sees the anti-commutator  $\sigma_{\hbar}$ , which coincides with the associative product on  $C^*(A)$  (cf. [22, 3.2]). This argument is closely related to the fact that the Jordan product  $\sigma_{\hbar}$  allows one to define functions of an observable, starting with  $A^2 \equiv A\sigma_{\hbar}A$ . Conversely, one could start with a squaring operation, and define the Jordan product by  $A\sigma_{\hbar}B = 1/2((A+B)^2 - A^2 - B^2)$ , cf. the Introduction of [9]. The connection between spectra and functions of observables is provided by the spectral calculus.

Next, we wish to relate the commutator  $\alpha_{\hbar}$  to the role observables play as generators of transformations of the space of pure states. As explained prior to (2.2), we may introduce states of an algebra of observables as normalized positive linear functionals  $\omega$  on  $\mathfrak{A}$ ; positivity here means that  $\omega(A^2) \geq 0$  for all  $A \in \mathfrak{A}$  (and  $A^2 = A\sigma_{\hbar}A$  as before), and normalized means that  $\|\omega\| = 1$  (which is equivalent to the property  $\omega(1) = 1$  if  $\mathfrak{A}$  has a unit 1, which is not the case for  $\mathfrak{A} = \mathcal{K}(\mathcal{H})$ ). The state space of  $\mathcal{K}(\mathcal{H})$  may be shown to be the space of all density matrices on  $\mathcal{H}$  (i.e., the positive trace-class operators [36] with unit trace). Pure states are as defined before, and we may consider the weak\*-closure  $P(\mathfrak{A})$  of the set of all pure states of  $\mathfrak{A}$ . In our example, any unit vector  $\psi \in L^2$  defines a pure state  $\omega_{\psi}$  by  $\omega_{\psi}(A) = (A\psi, \psi)$ , and, conversely, any pure state is obtained in this way. Noting that the space of pure states thus obtained is already weakly closed, we find that  $P(\mathcal{K}(\mathcal{H}))$  is equal to the projective Hilbert space  $P\mathcal{H}$  (which by definition is the set of equivalence classes  $[\psi]$  of vectors of unit length, under the equivalence relation  $\psi_1 \sim \psi_2$  if  $\psi_1 = \exp(i\alpha)\psi_2$  for some  $\alpha \in \mathbb{R}$ ). For example,  $P\mathbb{C}^2 = S^2$  (the two-sphere) is the pure state space of the algebra of hermitian  $2 \times 2$  matrices. More generally,  $P\mathcal{H}$  is a Hilbert manifold modeled on the orthoplement of an arbitrary vector in  $\mathcal{H}$ . Hence  $P\mathbb{C}^n$  is modeled on  $\mathbb{C}^{n-1}$ . To see this, take an arbitrary vector  $\chi \in \mathcal{H}$  (normalized to unity), and define a chart on the open set  $O_{\chi} \equiv \{\psi \in \mathcal{H} | (\psi, \chi) \neq 0\}$  by putting  $\Phi_{\chi} : O_{\chi} \rightarrow \chi^{\perp}$  equal to  $\Phi_{\chi}(\psi) = (\psi/(\psi, \chi)) - \chi$ . (We assume the inner product to be linear in the first entry.)

The fundamental point is that  $P\mathcal{H}$  has a Poisson structure [1]. To explain this, note first that  $T_{\psi}\mathcal{H} \simeq \mathcal{H}$ , since  $\mathcal{H}$  is a linear space; a vector  $\varphi \in \mathcal{H}$  determines a tangent vector  $\varphi_{\psi} \in T_{\psi}\mathcal{H}$  by its action on any  $f \in C^{\infty}(\mathcal{H})$

$$(\varphi_{\psi}f)(\psi) = \frac{d}{dt}f(\psi + t\varphi)|_{t=0}. \quad (3.7)$$

The symplectic form  $\omega$  on  $\mathcal{H}$  is then defined by

$$\omega(\varphi_{\psi}, \varphi'_{\psi}) = -2\hbar \text{Im}(\varphi, \varphi'). \quad (3.8)$$

We now regard  $A \in \mathfrak{A}$  not as an operator on  $\mathcal{H}$ , but as a function  $\tilde{f}_A$  on  $\mathcal{H}$ , defined on  $\psi \neq 0$  by

$$\tilde{f}_A(\psi) = \frac{(A\psi, \psi)}{(\psi, \psi)}. \quad (3.9)$$

(The value at  $\psi = 0$  is irrelevant). The point is that this definition quotients to  $P\mathcal{H}$ , so that  $A \in \mathfrak{A}$  defines a function  $f_A$  on  $P\mathcal{H}$  in the obvious way. Also, the symplectic structure quotients down to  $P\mathcal{H}$  (the professional way of seeing this [1] is that  $U(1)$  acts on  $\mathcal{H}$  by  $\psi \rightarrow \exp(i\alpha)\psi$ , this action is strongly Hamiltonian and leads to a moment map  $J : \mathcal{H} \rightarrow \mathbb{R}$  given by  $J(\psi) = (\psi, \psi)$ , and  $P\mathcal{H}$  is the Marsden-Weinstein reduction  $J^{-1}(1)/U(1)$ ), and this leads to the Poisson bracket

$$\{f_A, f_B\} = f_{A\alpha_{\hbar}B}, \quad (3.10)$$

with  $\alpha_{\hbar}$  defined in (3.5). An analogous equation determines the Poisson bracket on  $\mathcal{H}$  itself. As explained in (2.2) and below, the function  $\tilde{f}_A$  (hence  $A$ ) defines a vector field  $\tilde{X}_A$  on  $\mathcal{H}$ , whose value at the point  $\psi$  is found to be

$$\tilde{X}_A(\psi) = -\frac{i}{\hbar}A\psi. \quad (3.11)$$

The flow  $\tilde{\varphi}_t^A$  of this vector field is clearly

$$\tilde{\varphi}_t^A(\psi) = e^{-itA/\hbar}\psi. \quad (3.12)$$

Since this flow consists of unitary transformations of  $\mathcal{H}$ , it quotients to a flow  $\varphi_t^A$  on  $P\mathcal{H}$ , which is generated by a vector field  $X_A$  which is just the projection of  $\tilde{X}_A$  to the quotient space. This, in turn, is the vector field canonically related to  $f_A \in C^\infty(P\mathcal{H})$  via the Poisson structure (3.10).

Parallel to the discussion following (2.2), we remark that that  $\mathfrak{A}$  acts on  $\mathcal{H}$  irreducibly, in the sense that any two points in (a dense subset of)  $\mathcal{H}$  may be connected by some flow generated by an element of  $\mathfrak{A}$ . By projection, a similar statement holds for flows on  $P(\mathfrak{A}) = \mathfrak{P}\mathcal{H}$ . By (3.11), this is equivalent to the property that the collection  $\{A\psi | A \in \mathfrak{A}\}$  is dense in  $\mathcal{H}$  for each fixed  $\psi$ , and this, in turn, by (3.11) is exactly the irreducibility condition used in Definition 3 for Poisson algebras.

To sum up, we have shown that the product  $\alpha_{\hbar}$  indeed leads to the desired connection between observables and flows on the pure state space of  $\mathcal{A}_{\hbar}$  (note that all the  $\mathcal{A}_{\hbar}$  are isomorphic to  $\mathfrak{A}$  for  $\hbar \neq 0$ ), just as in the classical case.

Remarkably, the Jordan product  $\sigma_{\hbar}$  has a geometric expression in terms of the functions  $f_A$  on  $P\mathcal{H}$ , too [11]. Let  $g$  be the Kähler metric on  $\mathcal{H}$ , which is defined by (cf. (3.8))

$$g(\varphi_\psi, \varphi'_\psi) = \hbar \operatorname{Re}(\varphi, \varphi'). \quad (3.13)$$

Then a calculation shows that

$$f_{A\sigma_{\hbar}B} = \hbar g(\tilde{X}_A, \tilde{X}_B) + f_A f_B; \quad (3.14)$$

this should be compared with (3.10), which for this purpose may be rewritten as



$$f_{A\alpha_{\hbar}B} = \omega(\tilde{X}_A, \tilde{X}_B) + 0. \quad (3.15)$$

We see that the entire Jordan-Lie algebraic structure of  $\mathfrak{A}$  is encoded in the Kähler structure of  $P\mathcal{H}$ , which is given by hermitian metric  $\Omega$  defined by the inner product:

$$\Omega(\varphi_\psi, \varphi'_\psi) = \hbar(\varphi, \varphi'). \quad (3.16)$$

Clearly,  $\Omega = g - \frac{1}{2}i\omega$ .

### 3.2. Jordan-Lie algebras

We now generalize some of these considerations.

**DEFINITION 4.** *A real Banach space  $\mathfrak{A}$  equipped with two bilinear maps  $\sigma_{\hbar}, \alpha_{\hbar} : \mathfrak{A} \otimes_{\mathbb{R}} \mathfrak{A} \rightarrow \mathfrak{A}$ , which satisfy properties 1-7 in the preceding subsection, is called a Jordan-Lie algebra. If  $\hbar \neq 0$   $\mathfrak{A}$  is called non-associative, and if  $\hbar = 0$   $\mathfrak{A}$  is called associative. In the latter case the operation  $\alpha_0$  is only required to be densely defined.*

The Jordan-Lie structure of von Neumann's choice of  $\mathcal{B}(\mathcal{H})$  as the algebra of observables in quantum mechanics was emphasized in [17]. We here propose that Jordan-Lie algebras are the correct choice to take as algebras of observables in quantum mechanics; allowing more possibilities than  $\mathcal{B}(\mathcal{H})_{sa}$  or  $\mathcal{K}(\mathcal{H})_{sa}$  allows the incorporation of superselection rules, and the quantization of systems on topologically nontrivial phase spaces [26]. The example above already illustrates the remarkable fact that *conventional quantum mechanics may be described without the use of complex numbers*. The reader may object that a factor  $i$  appears in (3.5), but the resulting product  $\alpha_{\hbar}$  maps two self-adjoint operators into a self-adjoint operator, and it is the algebraic structure on  $\mathfrak{A}$  (given by  $\sigma_{\hbar}$  and  $\alpha_{\hbar}$ ), a real vector space, which determines all physical properties. Also, the (pure) state space is a real convex space and all observable numbers in quantum mechanics are of the form  $\omega(A)$ , where  $\omega$  is a state and  $A$  an observable.

A first major advantage of starting from Jordan-Lie algebras is that Poisson algebras are a special case (in which the symmetric product is associative), obtained by putting  $\hbar = 0$  in property 5. Hence classical and quantum mechanics are described by the same underlying algebraic structure (of which the former represents a limiting case), a point not at all obvious in the usual description in terms of either symplectic manifolds or Hilbert spaces.

A second comment is that the axioms imply that  $\mathfrak{A}$  must be the self-adjoint part of a  $C^*$ -algebra, so that we recover a mathematical structure that has proved to be exceptionally fruitful in the study of quantum mechanics [40, 26], quantum field theory [21], statistical mechanics [10, 21], and pure mathematics [14, 13, 32, 42, 43, 33]. Indeed, we may define an associative multiplication on  $\mathfrak{A}_{\mathbb{C}} = \mathfrak{A} \otimes_{\mathbb{R}} \mathbb{C}$  by means of

$$AB = A\sigma_{\hbar}B + \frac{1}{2}i\hbar A\alpha_{\hbar}B; \quad (3.17)$$

the associativity follows from the axioms, cf. [17]. The involution in  $\mathfrak{A}_{\mathbb{C}}$  is simply given by the extension of  $A^* = A$  for  $A \in \mathfrak{A}$ . The norm axioms imply that  $\mathfrak{A}_{\mathbb{C}}$  thus obtained is a  $C^*$ -algebra.

The meaning of  $\sigma_{\hbar}$  and  $\alpha_{\hbar}$  is the same as in the example of the compact operators. To explain this, it is convenient to use ‘Kadison’s function representation’ [25] of the self-adjoint part of any  $C^*$ -algebra (hence of any Jordan-Lie algebra). Let  $K$  be the state space of  $\mathfrak{A}$  (equipped with the weak\*-topology); this space is compact if  $\mathfrak{A}$  has a unit, which we shall assume (if not, one can adjoin one in a canonical way without any loss of information [43, 22]). Then  $\mathfrak{A}$  is isometrically isomorphic with the space  $A(K)$  of all affine real-valued continuous functions on  $K$  (with norm given by the supremum); since  $K$  is a convex subspace of the linear space of all continuous linear functionals on  $\mathfrak{A}$ , convex combinations  $\lambda(\omega_1) + (1 - \lambda)\omega_2$  ( $\lambda \in [0, 1]$ ) of states are well-defined, and a function  $f$  on  $K$  is called affine if  $f(\lambda(\omega_1) + (1 - \lambda)\omega_2) = \lambda f(\omega_1) + (1 - \lambda)f(\omega_2)$  for all  $\omega_i \in K$  and all  $\lambda \in [0, 1]$  (cf. [43, III.6] for detailed information on such spaces). The isomorphism between  $A \in \mathfrak{A}$  and  $\tilde{A} \in A(K)$  is simply given by  $\tilde{A}(\omega) = \omega(A)$ . The spectral theory of  $\mathfrak{A}$ , which, as we have seen in the case  $\mathfrak{A} = \mathcal{K}(\mathcal{H})_{\text{sa}}$ , is governed by the symmetric product  $\sigma_{\hbar}$  (using an argument which extends to the general case), translates into a spectral theory for such affine functions [4]. Conversely, if one starts from  $A(K)$  as the basic structure, one may set up a spectral calculus, which exploits the very special properties that  $K$  has because it is the state space of a  $C^*$ -algebra (hence, in particular, of a Jordan algebra). This spectral theory may then be used to *define*  $\sigma_{\hbar}$  [4], making the intimate connection between the symmetric product and the spectral calculus even clearer than in the realization of  $\mathfrak{A}$  as operators on a Hilbert space.

By the affine property, an element of  $A(K)$  is completely determined by its values on the pure state space  $P(\mathfrak{A})$  (which is the  $w^*$ -closure of the extreme boundary of  $K$  [43, 33]). We can define an equivalence relation  $\sim$  on  $P(\mathfrak{A})$ , saying that  $\omega_1 \sim \omega_2$  if both states give rise to unitarily equivalent representations (via the GNS construction, which provides a connection between states and representations [9, 43]). Each equivalence class defines a so-called folium of  $P(\mathfrak{A})$ . Each such folium is a Hilbert manifold, which is diffeomorphic (hence affinely isomorphic) to the pure state space  $P(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$  (cf. the previous subsection). Therefore, it admits a Poisson structure, which is defined exactly as in the case  $\mathfrak{A} = \mathcal{K}(\mathcal{H})$  (the compactness of  $A$  and  $B$  in (3.10) was not essential). The Poisson structures on the folia can be combined into a Poisson structure on  $P(\mathfrak{A})$ , which is degenerate iff  $\mathfrak{A}$  (unlike the compact operators) admits more than one equivalence class of irreducible representations. This eventually leads us to regard elements of  $A(K)$  (hence of  $\mathfrak{A}$ ) as generators of transformations of  $P(\mathfrak{A})$ , and we see that the flow of a given operator cannot leave a given folium. This suggests that  $P(\mathfrak{A})$  is a Poisson manifold, which is foliated by the symplectic leaves  $P(\mathcal{H})$ , but much remains to be done before this statement can be made precise, let alone proved (the main problems are to patch the folia together in the weak\*-topology on  $P(\mathfrak{A})$ , and to deal with

the states that are not pure but are weak\* limits of pure states. In the uniform topology on  $K$  and  $P(\mathfrak{A})$  things are easy, because  $P(\mathfrak{A})$  splits up as a collection of disjoint components, each component being a folium, but this topology is not the relevant one).

Thus the idea is to identify the inequivalent irreducible representations of  $\mathfrak{A}$  (that is, its superselection sectors [21]) with the symplectic leaves of the pure state space  $P(\mathfrak{A})$ , providing a nice parallel with the classical case. The total state space  $K$  of  $\mathfrak{A}$  may be equipped with a Poisson structure, too, but it is clear that the symplectic leaves of this Poisson space cannot be identified with inequivalent representations. For already in the simplest case where  $\mathfrak{A}$  consists of the hermitian  $n \times n$  matrices the state space is foliated by an uncountable number of symplectic leaves, whereas the inequivalent representations are labeled by a positive integer. (To see this, note that  $K$  can be embedded in the dual  $\mathfrak{u}(n)^*$  of the Lie algebra of  $U(n)$ , equipped with the canonical Lie-Poisson structure, and this embedding is a Poisson morphism. Hence the symplectic leaves of  $K$  are simply given by those leaves of  $\mathfrak{u}(n)^*$  which lie in  $K$ ; these are generalized flag manifolds, and there are uncountably many even of a given orbit type).

In any case, we see that the role of the antisymmetric product  $\alpha_{\hbar}$  as the agent which relates observables to flows on the pure state space survives unscratched for Jordan-Lie algebras. Conversely, we would like to define this product in terms of the Poisson structure on  $P(\mathfrak{A})$ . This can presumably be done using a result of Shultz [41], who proved that the commutator on the self adjoint part  $\mathfrak{A}$  of a  $C^*$ -algebra  $\mathfrak{A}$  is *abstractly* determined by specifying transition probabilities and an orientation on  $P(\mathfrak{A})$ . These transition probabilities are the usual ones if one passes from states to their GNS representations (and are zero for disjoint states, that is, states leading to inequivalent representations). Specifying  $|(\psi_1, \psi_2)|^2$  plus an orientation is equivalent to specifying  $\text{Im}(\psi_1, \psi_2)$ , so we see from (3.8) that the theorem in [41] can very simply be understood by saying that the commutator is given by the Poisson bracket (3.10), and that Poisson and Jordan isomorphisms between two state spaces are induced by isomorphisms of the corresponding Jordan-Lie algebras.

We return to the axioms 1-7 on a Jordan-Lie algebra. Especially the norm axioms, but also property 5 look rather arbitrary, and it would be nice to reformulate them in such a way, that the following question may be answered: *which physical postulates of quantum mechanics imply its description in terms of Jordan-Lie algebras and their state spaces?* A similar question concerned with the Hilbert space formulation of conventional quantum mechanics is analyzed in [30, 7]. Since a Jordan-Lie algebra is isomorphic to the self-adjoint part of a  $C^*$ -algebra, we can look at the literature for help. It turns out to be fruitful to shift emphasis from the Jordan-Lie algebra  $\mathfrak{A}$  to its state space  $K$  (from which  $\mathfrak{A}$  can be recovered as  $A(K)$ , as we have reviewed above). The question above may then be reformulated by asking which properties of a compact convex set  $K$  make  $A(K)$  isomorphic to a Jordan-Lie algebra, and what the physical meaning of these properties is (as before, we stress the point that

by eliminating complex numbers and Hilbert spaces from quantum mechanics through its reformulation in terms of Jordan-Lie algebras and their state spaces, we feel that we have come closer to the physical meaning of this theory).

The latter question has partly been answered in the work of Alfsen and Shutz [4, 3, 41], and others (cf. the reviews [2, 45]). As a consequence of these papers, the origin of the Jordan structure in quantum mechanics (as well as the norm axioms, which only use the Jordan product  $\sigma_{\hbar}$ ) is now quite well understood. The key property of  $K$  that leads to a Jordan structure and the associated spectral calculus is the existence of sufficiently many projective faces in  $K$ ; a projective face plays a role similar to that of a closed subspace of a Hilbert space (or the corresponding projector) and is physically a yes-no question. Projective faces are orthocomplemented, and have other nice properties making them suitable as a basic ingredient of quantum logic [7, 12]. Other properties of  $K$  which are necessary to derive the Jordan structure are related to the property that pure states in quantum mechanics can be prepared through filtering procedures, and to the symmetry of transition amplitudes (which reflects the symmetry between pure states and finest detectors [21]).

Further properties of the state space  $K$  leading to a Lie bracket on  $\mathfrak{A} \simeq \mathfrak{A}(\mathfrak{K})$  are known [3, 2], but their physical meaning is not so clear. We hope to be able to show that these properties are equivalent to  $P(\mathfrak{A})$  admitting a Poisson structure which foliates the pure state space in a way consistent with the representation theory of  $\mathfrak{A}$  as a Jordan algebra. A crucial property of non-associative Jordan-Lie algebras (i.e.,  $\hbar \neq 0$ ) is that the restriction of  $A(K)$  to  $P(\mathfrak{A})$  does not coincide with the space of all continuous functions on  $P(\mathfrak{A})$  (unlike the classical case; the essential point is that not nearly every function on  $P(\mathfrak{A})$  extends to an affine function on  $K$ , because non-pure elements of  $K$  generically have many decompositions as convex sums of pure states [9, 43]). This non-uniqueness constrains the allowed functions on the extreme boundary  $P(\mathfrak{A})$  of  $K$ , which do have an affine extension to  $K$ , enormously. Such constraints do not arise when every mixed state in  $K$  has a unique extremal decomposition, and this happens precisely when  $\mathfrak{A}$  is associative, i.e., in the classical case.). Together with the Poisson structure this property should be related to the uncertainty principle (at least in its naive textbook formulation).

### 3.3. Representation theory of Jordan-Lie algebras

As we have seen in Definition (2), a representation of a Poisson algebra is a map into  $C^\infty(S)$  for some symplectic space  $S$ , which preserves all the algebraic structures and in addition satisfies a completeness condition. The motivation was that  $C^\infty(S)$  for symplectic  $S$  is a ‘canonical’ model of a Poisson algebra. More importantly, irreducibility implies that  $S$  must be symplectic. Similarly, the canonical model of a Jordan-Lie algebra is the algebra of all bounded self-adjoint operators on a complex Hilbert space  $\mathcal{H}$ . The latter is naturally equipped with the Jordan-Lie structure (3.5), and this motivates

DEFINITION 5. A representation of a non-associative Jordan-Lie algebra  $\mathfrak{A}$  is a map  $\pi_q^\chi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_\chi)$ , for some Hilbert space  $\mathcal{H}_\chi$ , satisfying for all  $A, B \in \mathfrak{A}$

1.  $\pi_q^\chi(\lambda A + \mu B) = \lambda \pi_q^\chi(A) + \mu \pi_q^\chi(B)$  (linearity);
2.  $\pi_q^\chi(A\sigma_\hbar B) = \frac{1}{2}(\pi_q^\chi(A)\pi_q^\chi(B) + \pi_q^\chi(B)\pi_q^\chi(A))$  (preserves Jordan product);
3.  $\pi_q^\chi(A\alpha_\hbar B) = \frac{1}{i\hbar}(\pi_q^\chi(A)\pi_q^\chi(B) - \pi_q^\chi(B)\pi_q^\chi(A))$  (preserves Lie product);
4.  $\pi_q^\chi(A)^* = \pi_q^\chi(A)$  (self-adjointness).

These conditions are, of course, equivalent to the usual ones on representations of the  $C^*$ -algebra  $\mathfrak{A}_\mathbb{C}$  (the self-adjointness condition 4 is the requirement that  $\pi_q^\chi$  is a  $*$ -representation of  $\mathfrak{A}_\mathbb{C}$ ), but we have put them in the given form to make the analogy with the classical Definition 2 clear. In similar vein, the classical irreducibility condition Definition 3 is, as we have seen from the discussion following (3.12), essentially the same as the usual definition of irreducibility for representations of  $C^*$ -algebras, which in the present framework reads

DEFINITION 6. A representation  $\pi_q^\chi$  of a Jordan-Lie algebra  $\mathfrak{A}$  on a Hilbert space  $\mathcal{H}_\chi$  is called irreducible iff every vector in  $\mathcal{H}_\chi$  is cyclic for  $\pi_q^\chi(\mathfrak{A})$  (that is, the set  $\{A\psi | A \in \mathfrak{A}\}$  is dense in  $\mathcal{H}_\chi$  for each fixed  $\psi \in \mathcal{H}_\chi$ ).

All this may be reformulated in terms of the (pure) state space of  $\mathcal{H}_\chi$ , and the Jordan and Lie products on  $A(K)$  as discussed in the previous subsection, but we leave this to the reader.

There is a decisive difference between the classical case ( $\hbar = 0$ ; Jordan product  $\sigma \equiv \sigma_0$  associative) and the quantum case as far as irreducibility is concerned. Irreducible representations of a Poisson algebra  $C^\infty(M)$  are highly reducible as representations of the corresponding Jordan algebra (in which the anti-symmetric product  $\alpha$  is ignored), whereas irreducible representations of this Poisson algebra (which are just points of  $M$ ) do not lead to representations of  $C^\infty(M)$  at all. In the quantum case, a representation of a non-associative Jordan-Lie algebra is irreducible iff it is irreducible as a representation of the underlying Jordan algebra. This looks curious, because the irreducibility condition above may be formulated in terms of the vector fields (3.11), which are defined using the Lie product (see (3.10)). However, the unitary flow (3.12) is completely defined in terms of the Jordan product (which allows the definition of functions of an operator).

The naive quantum analogue of the generalized moment map  $J$  (cf. Theorem 1) is rather trivial: given a representation  $\pi_q^\chi(\mathfrak{A})$ , we may define a map  $\tilde{J} : \mathcal{H}_\chi \rightarrow K$  (where  $K$  is the state space of  $\mathfrak{A}$ ) by specifying the value of the state  $\tilde{J}(\psi)$  on arbitrary  $A \in \mathfrak{A}$  to be

$$(\tilde{J}(\psi))(A) = \frac{(\pi_q^\chi(A)\psi, \psi)}{(\psi, \psi)}. \quad (3.18)$$

This evidently reduces to a map  $J : P\mathcal{H}_\chi \rightarrow K$ , which is the naive quantum analogue of the classical generalized moment map. Namely, for  $\pi_q^\chi$  irreducible, the image of  $J$  is contained in the pure state space  $P(\mathfrak{A})$ . Thus we see that the quantum moment map just expresses the correspondence between states and vectors in a Hilbert space, which is central to the GNS construction [43, 9], and lies at the heart of operator algebras. A difference between the classical and the naive quantum moment map is that the image of the former is the set of pure states, even if the representation is reducible, while the image of the latter may well lie among the mixed states (namely if the representation is reducible). Also, the Marsden-Weinstein symplectic reduction construction [1, 29] cannot be ‘quantized’ in terms of  $\tilde{J}$  in any obvious way. Hence one needs a deeper quantum analogue of the classical moment map, and this is given by the concept of a Hilbert  $C^*$ -module, see [28].

The quantum counterpart of the classical Theorem 2 has yet to be proved (and even properly formulated); this would express that  $P(\mathfrak{A})$  is foliated by its symplectic leaves, which, as we have seen in the preceding subsection, should be identified with folia of states leading to equivalent representations.

#### 3.4. The group algebra

For reasons to emerge later, a quantum analogue of the Poisson algebra  $C^\infty(\mathfrak{g}^*)$  (cf. subsect. 2.3) is the group algebra  $JL(G) = C^*(G)_{\text{sa}}$ ; it is the quantum algebra of observables of a particle whose only degree of freedom is internal. Here  $G$  is any Lie group with Lie algebra  $\mathfrak{g}$ . For simplicity, we only define  $JL(G)$  for unimodular  $G$  (look up  $C^*(G)$  in [33] for the general case). The starting point is to construct a dense subalgebra of  $C^*(G)$ . This is done by defining a product  $*$  and involution  $\bar{\phantom{x}}$  on  $C_c^\infty(G)$  by

$$(f * g)(x) = \int_G dx f(xy)g(y^{-1}); \quad f^*(x) = \overline{f(x^{-1})}, \quad (3.19)$$

where  $dx$  is a Haar measure on  $G$ . The norm is defined in [33]; in the special case that  $G$  is amenable (this holds, for example, when  $G$  is compact) one may put  $\|f\| = \|\pi_q^L(f)\|$ , where  $\pi_q^L$  is a representation of  $C_c^\infty(G)$  (regarded as an associative  $*$ -algebra) on  $\mathcal{H}_L = L^2(G)$ , given by

$$(\pi_q^L(f)\psi)(x) = \int_G dy f(y)(\pi_L(y)\psi)(x), \quad (3.20)$$

with  $(\pi_L(y)\psi)(x) = \psi(y^{-1}x)$ . The closure of  $C_c^\infty(G)$  in this norm is the group algebra  $C^*(G)$ . The corresponding Jordan-Lie algebra  $JL(G)$  is its self-adjoint part, equipped with the products  $\sigma_\hbar$  and  $\alpha_\hbar$ , defined as in (3.5) (with  $AB$  replaced by  $f * g$ , etc.).

The representation theory of  $JL(G)$  coincides with that of  $C^*(G)$ , which is well-known [33]: every (non-degenerate) representation  $\pi_q^\chi$  of  $JL(G)$  on a Hilbert space  $\mathcal{H}_\chi$  corresponds to a unitary representation  $\pi_\chi$  of  $G$  on  $\mathcal{H}_\chi$ , the passage from  $\pi_\chi(G)$  to  $\pi_q^\chi(JL(G))$  being accomplished by the analogue

of (3.20), with  $L$  replaced by  $\chi$ . In particular, irreducible representations of  $JL(G)$  correspond to irreducible unitary representations of  $G$ .

In traditional quantization theory (applied to this special case) one tried to associate a Hilbert space and certain operators to a co-adjoint orbit  $\mathcal{O} \subset \mathfrak{g}^*$  and the associated Poisson algebra  $C^\infty(\mathcal{O})$  (which we look upon as an irreducible representation of  $C^\infty(\mathfrak{g}^*)$ ). This was very successful in special situations, e.g.,  $G$  nilpotent. In that case there is a one-to-one correspondence between co-adjoint orbits and unitary representations, given by the Dixmier-Kirillov theory [15]. The same strategy was reasonably successful in some other cases, like  $G$  compact and semi-simple, when any irreducible unitary representation of  $G$  can be brought into correspondence with at least some co-adjoint orbit via the Borel-Weil theory [24]; on the other hand, most co-adjoint orbits do not correspond to any unitary representation of  $G$  at all. However, in the general case no correspondence between co-adjoint orbits and irreducible representations exists, and modern research in representation theory looks in different directions [46] (note that this does not undermine the hard fact that the classical irreducible representations of  $C^\infty(\mathfrak{g}^*)$  are completely classified by the co-adjoint orbits and their covering spaces).

The natural correspondence between classical and quantum mechanics exists at an algebraic level, namely in their respective Jordan-Lie algebras of observables. The irreducible representations of a classical Poisson algebra are not necessarily related to those of the corresponding quantum Jordan-Lie algebra, and both should be constructed in their own right.

#### 4. QUANTIZATION

##### 4.1. The definition of a quantization

We now return to the Weyl quantization on  $\mathbb{R}^n$  reviewed in subsect. 3.1. We have seen how we may regard  $Q_\hbar$  as a map from the dense subspace  $\overline{\mathfrak{A}_0}$  of the commutative Banach algebra  $\mathfrak{A}_0 = \mathcal{C}_0(\mathcal{T}^*\mathbb{R}^n)$  to the space of self-adjoint compact operators  $\mathfrak{A} = \mathcal{K}(\mathcal{L}^2(\mathbb{R}^n))_{\text{sa}}$ . Here  $\mathfrak{A}_0$  also has a densely defined Poisson structure (which is, in particular, defined on  $\overline{\mathfrak{A}_0}$ ), and may be regarded as an associative Jordan-Lie algebra, equipped with the products  $\cdot = \sigma \equiv \sigma_0$  and  $\{ , \} = \alpha \equiv \alpha_0$ . The space  $\mathfrak{A}$  may be dressed up with the products  $\sigma_\hbar$  and  $\alpha_\hbar$ , defined in (3.5), and thus a family of non-associative Jordan-Lie algebras  $\{\mathcal{A}_\hbar\}$  is defined (the norm in  $\mathcal{A}_\hbar$  is borrowed from  $\mathfrak{A}$ , and is independent of  $\hbar$  for  $\hbar \neq 0$ ). The norm on  $\mathfrak{A}_0$  is defined in (3.4). We define  $Q_0 : \mathfrak{A}_0 \rightarrow \mathfrak{A}_0$  as the identity map. It may be shown [27] that the following properties hold for all  $f, g \in \overline{\mathfrak{A}_0}$ :

1.  $\lim_{\hbar \rightarrow 0} \| Q_\hbar(f)\sigma_\hbar Q_\hbar(g) - Q_\hbar(f\sigma_0 g) \| = 0$ ;
2.  $\lim_{\hbar \rightarrow 0} \| Q_\hbar(f)\alpha_\hbar Q_\hbar(g) - Q_\hbar(f\alpha_0 g) \| = 0$ ;
3. the function  $\hbar \rightarrow \| Q_\hbar(f) \|$  is continuous on  $I = \mathbb{R}$ .

Condition 2 is an analytic reformulation of the correspondence between commutators of operators and Poisson brackets of functions first noticed by Dirac.

The first condition is based on the correspondence between anti-commutators of operators and pointwise products of functions, first noticed by von Neumann. The third condition is a precise formulation of (one form) of the correspondence principle due to Bohr. Recalling that  $f\sigma_0g = fg$  and  $f\alpha_0g = \{f, g\}$ , note the consistency of the above conditions with (3.14) and (3.10). In the context of  $C^*$ -algebras conditions 2 and 3 in their present form were first written down by Rieffel [38] (who did not impose either condition 1 or self-adjointness on a quantization map). The connection between deformations of algebras and quantization theory was analyzed in a different mathematical setting in [8, 6].

The example of a particle on  $\mathbb{R}^n$  and the general considerations in sections 2 and 3 motivate the following

**DEFINITION 7.** *Let  $\mathfrak{A}_0$  be a commutative Jordan algebra with a densely defined Poisson bracket (making  $\mathfrak{A}_0$  into an associative Jordan-Lie algebra, cf. Def. 4), and let  $\overline{\mathfrak{A}_0}$  be a dense subalgebra on which the Poisson bracket is defined. A quantization of this structure is a family  $\{A_{\hbar}\}_{\hbar \in I}$  of non-associative Jordan-Lie algebras (Def. 4), and a family  $\{Q_{\hbar}\}_{\hbar \in I}$  of maps defined on  $\overline{\mathfrak{A}_0}$ , such that the image of  $Q_{\hbar}$  is in  $A_{\hbar}$ , and the above conditions 1-3 are satisfied.*

As we have seen, Weyl quantization satisfies this definition. A generalization of Weyl quantization to arbitrary Riemannian manifolds is given in [27]. The axioms above are not quite satisfied by this generalized quantization prescription, in that the range in  $\hbar$  for which  $Q_{\hbar}$  is defined depends on its argument. This is easily remedied, however, by constructing cutoff functions in  $\hbar$ , cf. the example below. The cutoff, on the other hand, upsets the physical interpretation of  $Q_{\hbar}(f)$  as the quantum observable corresponding to the classical observable  $f$  for all  $\hbar \in I$ , and for that reason in [27] we preferred to leave  $Q_{\hbar}(f)$  undefined whenever it could no longer be interpreted properly. This complication only occurs for manifolds for which the exponential map is not a diffeomorphism on the entire tangent space at each point. A further generalization is to admit internal degrees of freedom, through which the particle can couple to a gauge field. This case is covered in [27], too, and from this general class of examples it has become clear that the definition of quantization given above is satisfied by a number of realistic physical examples.

A non-self-adjoint version of the quantization of  $C_0(\mathfrak{g}^*)$  by  $C^*(G)$  (cf. subsects. 2.3 and 3.4) was first given by Rieffel [39], and the physically relevant self-adjoint version, i.e., the construction of the maps  $Q_{\hbar} : C_0(\mathfrak{g}^*) \rightarrow JL(G)$  is a special case of the theory in [27] if  $G$  is compact (obtained by taking  $P = H = G$  in that paper, and exploiting the fact that  $(T^*G)/G \simeq \mathfrak{g}^*$  with the usual Poisson structure). We define  $\overline{\mathfrak{A}_0} \subset \tilde{C}_0(\mathfrak{g}^*)$  as the space of those functions  $f$  on  $\mathfrak{g}^*$  whose Fourier transform  $\check{f}$  is in  $C_c^\infty(\mathfrak{g})$  (since  $\mathfrak{g}^* \simeq \mathbb{R}^n$  we can define the Fourier transform as usual, cf. (3.1), omitting the  $x$ -dependence). The quantization map is given by

$$(Q_{\hbar}(f))(e^{-\hbar X}) = \hbar^{-n} \check{f}(X), \quad (4.1)$$



which defines the left-hand side as an element of  $C^*(G)_{\text{sa}} = JL(G)$  for those values of  $\hbar$  for which  $\hbar$  times the support of  $\check{f}$  lies in the neighbourhood of  $0 \in \mathfrak{g}$  on which the exponential function is a diffeomorphism from  $\mathfrak{g}$  to  $G$ . Since  $\check{f}$  has compact support, the allowed values of  $\hbar$  will lie in an interval  $(-\hbar_0, \hbar_0)$ , where  $\hbar_0$  depends on  $f$ . If the group  $G$  is exponential (which is the case if  $G$  is simply connected and nilpotent [15]) then  $\hbar_0 = \infty$ . In general, one could extend the quantization to any value of  $\hbar$ , without violating the conditions required by Def. 7, by multiplying  $Q_\hbar(f)$  by a function  $h$  which is 1 in  $(-.99\hbar_0, .99\hbar_0)$  (say).

#### 4.2. Positivity and continuity

While the Weyl quantization of subsect. 3.1 (as well as its generalization to Riemannian manifolds) satisfies Def. 7 of a quantization, there are two serious problems with it. The first is lack of positivity; this means that if  $f \geq 0$  in  $\mathfrak{A}_0 = \mathfrak{C}_0(\mathfrak{T}^*\mathbb{R}^n)$  then it is not necessarily true that  $Q_\hbar(f) \geq 0$  in  $\mathfrak{A}$  (see e.g. [18, 2.6]). From the equality

$$(Q_\hbar(f)\Omega, \Omega) = \int_{T^*\mathbb{R}^n} \frac{d^n x d^n p}{(2\pi)^n} W_\Omega^\hbar(x, p) f(x, p), \quad (4.2)$$

with the Wigner function

$$W_\Omega^\hbar(x, p) = \int_{\mathbb{R}^n} d^n \dot{x} e^{ip\dot{x}} \Omega(x - \frac{1}{2}\hbar\dot{x}) \overline{\Omega(x + \frac{1}{2}\hbar\dot{x})}, \quad (4.3)$$

we see that the potential non-positivity of  $Q_\hbar(f)$  is equivalent to the fact that the Wigner distribution function (4.3) is not necessarily positive definite.

The second problem is that  $Q_\hbar$  (for fixed  $\hbar \neq 0$ ) is not continuous as a map from  $\overline{\mathfrak{A}_0}$  to  $\mathfrak{A}$  (both equipped with their respective norm topologies). Hence it cannot be extended to  $\mathfrak{A}_0$  in any natural way. The problem here is that we wish to work in a Banach-algebraic framework; the map  $Q_\hbar$  is continuous as an operator from  $L^2(T^*\mathbb{R}^n)$  to  $HS(L^2(\mathbb{R}^n))$  (if both are regarded as Hilbert spaces, the latter being the space of Hilbert-Schmidt operators on  $L^2(\mathbb{R}^n)$ , with the inner product  $(A, B) = \text{Tr}AB^*$ ), and also as a map from the Schwartz space  $\mathcal{S}'(T^*\mathbb{R}^n)$  to the space of continuous linear maps from  $\mathcal{S}(\mathbb{R}^n)$  to  $\mathcal{S}'(\mathbb{R}^n)$ , cf. [18, 19].

Both problems may be resolved simultaneously if we construct a positive quantization, that is, find a map  $Q'_\hbar : \mathfrak{A}_0 \rightarrow \mathfrak{A}$  which is positive. For a positive map between two  $C^*$ -algebras is automatically continuous (see [43], p. 194). Let  $\{\Omega_\hbar\}_{\hbar>0}$  be a family of normalized vectors in  $L^2(\mathbb{R}^n)$ , which satisfy the condition that in the limit  $\hbar \rightarrow 0$  the Wigner function  $W_{\Omega_\hbar}^\hbar$  is smooth in all variables (including  $\hbar$ ), vanishes rapidly at infinity, and converges to  $(2\pi)^n \delta(x, p)$  in the distributional topology defined by the test function space  $\overline{\mathfrak{A}_0}$  (defined after (3.1)). An example is

$$\Omega_\hbar(x) = (\pi\hbar)^{-n/4} e^{-x^2/2\hbar}, \quad (4.4)$$

with Wigner function

$$W_{\Omega_h}^h(x, p) = (2/\hbar)^n e^{-(x^2+p^2)/\hbar}. \quad (4.5)$$

We then define a new quantization map  $Q_h^\Omega$  by

$$Q_h^\Omega(f) = Q_h(W_{\Omega_h}^h * f), \quad (4.6)$$

with  $Q_h$  the Weyl quantization (3.3),  $W_{\Omega_h}^h$  defined by  $W_{\Omega_h}^h(x, p) = W_{\Omega_h}^h(-x, -p)$ , and  $*$  being the convolution product in  $\mathbb{R}^{2n}$ . It follows from Prop. 1.99 in [18] that  $Q_h^\Omega$  is a positive map. Since the uniform operator norm is majorized by the Hilbert-Schmidt norm, it follows from the triangle inequality and the first continuity property of  $Q_h$  mentioned above that  $Q_h^\Omega$  defines a quantization if for all  $f, g \in \overline{\mathfrak{A}_0}$

$$\begin{aligned} L^2 - \lim_{\hbar \rightarrow 0} \left( \{W_{\Omega_h}^h * f, W_{\Omega_h}^h * g\} - W_{\Omega_h}^h * \{f, g\} \right) &= 0; \\ L^2 - \lim_{\hbar \rightarrow 0} \left( (W_{\Omega_h}^h * f) \cdot (W_{\Omega_h}^h * g) - W_{\Omega_h}^h * (f \cdot g) \right) &= 0, \end{aligned} \quad (4.7)$$

and if the function  $\hbar \rightarrow \|Q_h^\Omega(f)\|$  is continuous for all such  $f$ . These conditions are all satisfied if  $\Omega_h$  is as specified prior to (4.4), and thus  $Q_h^\Omega$  is indeed a positive definite quantization (note that  $Q_h^\Omega$  is automatically self-adjoint, since  $W_{\Omega_h}^h$  is real-valued). It can be extended to  $\mathfrak{A}_0$  by continuity, but the extension obviously does not satisfy the quantization condition involving the Poisson bracket (which is not a continuous map on  $\mathfrak{A}_0$  in either variable).

This procedure may be extended to arbitrary manifolds  $Q$ ; the smearing  $f \rightarrow W_{\Omega_h}^h * f$  will in general be replaced by the use of Friedrichs mollifiers. It is clear that this positive definite quantization procedure is not intrinsic: it depends on the choice of the  $\Omega_h$ . It may be argued that the Weyl quantization procedure is not intrinsic either, because from a geometric point of view [27] it relies on the choice of a diffeomorphism between a tubular neighbourhood of  $Q$  in  $TQ$ , and one of  $\Delta Q$  in  $Q \times Q$ . In any case, one may argue that points in space should be stochastic objects, with a probability distribution related to  $\Omega_h$ . This point of view is defended, in a quite different context, in [35, 5].

## 5. LIE GROUPOIDS, LIE ALGEBROIDS, AND THEIR JORDAN-LIE ALGEBRAS

The (generalized [27]) Weyl quantization of  $C_0(T^*Q)$  by  $\mathcal{K}(L^2(Q))_{\text{sa}}$  and the quantization of  $C_0(\mathfrak{g}^*)$  by  $JL(G) = C^*(G)_{\text{sa}}$  are both special cases of a rather general construction involving Lie groupoids, which are a certain generalization of Lie groups that are of great physical and mathematical relevance (cf. [31, 16] for a comprehensive discussion of these structures, illustrated with many examples).

### 5.1. Basic definitions

We recall that a category  $G$  is a class  $B$  of objects together with a collection of arrows. Each arrow  $x$  leads from object  $s(x)$  (the source of the arrow) to the

object  $t(x)$  (the target). If  $s(x) = t(y)$  then the composition  $xy$  is defined as an arrow from  $s(y)$  to  $t(x)$ , and this partial multiplication on  $G$  is associative whenever it is defined. Also, each object  $b \in B$  comes with an arrow  $i(b)$ , which serves as the identity map from  $s(i(b)) = b$  to  $t(i(b)) = b$ , so that  $xi(b) = x$  (defined when  $s(x) = b$ ) and  $i(b)x = x$  (defined when  $t(x) = b$ ). Hence we obtain an inclusion  $i$  of  $B$  into  $G$ . A category is called small if  $B$  is a set.

DEFINITION 8. *A groupoid is a small category in which each arrow is invertible.*

Hence for each  $x \in G$  the arrow  $x^{-1}$  is defined, with  $s(x^{-1}) = t(x)$  and  $t(x^{-1}) = s(x)$ , and one has  $i \circ s(x) = x^{-1}x$  and  $i \circ t(x) = xx^{-1}$ . We may regard  $G$  as a fibered space over  $B$ , with two projections  $S : G \rightarrow B$  and  $t : G \rightarrow B$ . One may pass to topological groupoids by requiring continuity of the relevant structures, and to Lie groupoids by demanding smoothness:

DEFINITION 9. *A Lie groupoid is a groupoid in which  $G$  and  $B$  are smooth manifolds (taken to be Hausdorff, paracompact and finite-dimensional), so that the inclusion  $i$  is a smooth embedding, the projections  $s$  and  $t$  are smooth surjective submersions, and the inverse  $x \rightarrow x^{-1}$ , as well as the partial multiplication  $(x, y) \rightarrow xy$  are smooth maps.*

Variations on this definition are possible, cf. [31, 16]; for example, in the former ref. the assumption is added that  $G$  is transitive, in the sense that the map  $s \times t : G \rightarrow B \times B$  is surjective (that is, any two points in  $B$  can be connected by an arrow), but since a corresponding transitivity assumption is not part of the definition of a Lie algebroid (see below) in [31], we follow [16] in dropping it.

We see that a Lie group is a special case of a Lie groupoid, namely a case in which  $B$  consists of one point  $b$  (and  $i(b) = e$  is the identity of  $G$ ), so that all arrows can be composed. One may generalize the passage from a Lie group to a Lie algebra in the present context. First note that each  $x \in G$  defines not only an arrow from  $s(x)$  to  $t(x)$ , but in addition leads to a map  $L_x : t^{-1}(s(x)) \rightarrow t^{-1}(t(x))$ , defined by  $L_x(y) = xy$ . Similarly, one has a map  $R_x : s^{-1}(t(x)) \rightarrow s^{-1}(s(x))$  given by  $R_x(y) = yx$ . As in the Lie group case, we would like to define left- and right invariant vector fields on  $G$ . Hence we would obtain (say) a left-invariant flow  $\varphi_\tau$  on  $G$ , satisfying  $\varphi_\tau(L_x(y)) = L_x\varphi_\tau(y)$  for  $y \in t^{-1}(s(x))$ . The problem is that  $L_x$  is only a partially defined multiplication, so that the right-hand side is only defined if  $t(\varphi_\tau(y)) = s(x)$ , that is, the target of  $\varphi_\tau(y)$  must not depend on the time  $\tau$ . Hence  $(d/d\tau)t(\varphi_\tau) = 0$ , or  $t_*X = 0$  if  $X = (d/d\tau)(\varphi_\tau)|_{\tau=0}$ . In conclusion, we may define a left-invariant vector field  $\xi_L$  by the conditions

$$t_*\xi_L = 0; \quad (L_x)_*\xi_L = \xi_L \quad \forall x \in G, \quad (5.1)$$

and a right-invariant vector field  $\xi_R$  by the conditions

$$s_*\xi_R = 0; \quad (R_x)_*\xi_R = \xi_R \quad \forall x \in G. \quad (5.2)$$

It is easily shown [31, 16] that the commutator (Lie bracket) of two left (right) invariant vector fields is left (right) invariant. Hence we may define

**DEFINITION 10.** *The Lie algebroid  $\mathfrak{g}$  of a Lie groupoid  $G$  is the real vector space of all vector fields on  $G$  satisfying (5.1), equipped with the following structures:*

- i) a projection  $pr : \mathfrak{g} \rightarrow B$  (namely the obvious one, coming from the projections  $TG \rightarrow G \xrightarrow{s} B$ ), which makes  $\mathfrak{g}$  a vector bundle over  $B$ ;*
- ii) a projection  $q : \mathfrak{g} \rightarrow TB$ , given by  $q = s_*$ ;*
- iii) a Lie bracket on  $\Gamma(\mathfrak{g})$  (the space of smooth sections of  $\mathfrak{g}$ ), which is given by the commutator on  $\Gamma(TG)$ , and which satisfies*

$$q([\xi_L^1, \xi_L^2]) = [q(\xi_L^1), q(\xi_L^2)]; \quad (5.3)$$

$$[\xi_L^1, f\xi_L^2] = f[\xi_L^1, \xi_L^2] + q(\xi_L^1)f \cdot \xi_L^2 \quad \forall f \in C^\infty(B). \quad (5.4)$$

Of course, an equivalent definition is obtained by replacing (5.1) by (5.2), and  $s$  and  $s_*$  by  $t$  and  $t_*$ , respectively. One may define a Lie algebroid without reference to Lie groupoids as vector bundle  $E$  over  $B$ , together with an additional projection  $q : E \rightarrow TB$  satisfying (5.3) (the ‘anchor’ of  $E$  [31]) and a Lie bracket on  $\Gamma(E)$  satisfying the analogue of (5.4). A Lie algebroid is called transitive if  $q$  is a surjective; if a Lie groupoid  $G$  is transitive then so is its algebroid  $\mathfrak{g}$ . A simple example is  $E = TQ$  as a vector bundle over  $Q$ , with  $q$  the identity map.

One may generalize the identification  $\mathfrak{g} \simeq T_e G$  for a Lie group  $G$  as follows. Since  $x = x(x^{-1}x) = L_x(i \circ s(x))$ , every left-invariant vector field  $\xi_L$  on  $G$  is determined by its values at  $i(B) \equiv G_0 \subset G$ . We have the decomposition  $T_{G_0}G = T_{G_0}G_0 \oplus \ker(t_*)|_{T_{G_0}G}$  (where  $|$  means ‘restricted to’), so we see that  $\mathfrak{g} \simeq T_{G_0}G/T_{G_0}G_0$ , which is just the normal bundle  $N_i$  of the embedding  $i : B \rightarrow G$ . Equivalently, if we define  $T^t G$  as the vector bundle over  $G$  consisting of elements of  $TG$  annihilated by  $t_*$  (with the canonical projection  $pr_t$  onto  $G$  borrowed from  $TG$ ), then  $\mathfrak{g} = i^*(T^t G)$ , the pull-back bundle over  $B$  given by the map  $i : B \rightarrow G$ . Conversely,  $T^t G = s^*(\mathfrak{g})$  as a pull-back bundle [31]. Note that  $T^t G$  is itself a Lie algebroid over  $G$ , with the anchor  $q : T^t G \rightarrow TG$  just given by inclusion.

An interesting property of a Lie algebroid  $E$  is that a connection on  $E$  allows one to define generalized geodesics on  $E$  (hence on the base space  $B$ ). Namely, one obtains a vector field  $\xi$  on  $E$ , whose value at  $Y \in E$  is given by the horizontal lift of  $q(Y) \in TB$  at  $Y$ . The flows of this field are the desired generalized geodesics (for  $E = TB$  equipped with the Levi-Civita connection these are the usual geodesics). This leads to the construction of a map  $\exp : \mathfrak{g} \rightarrow G$  which generalizes the one for Lie groups [34]. Namely, by the preceding paragraph  $T^t G$  regarded as a vector bundle over  $G$  inherits the chosen connection  $A$  on  $\mathfrak{g}$  (considered a vector bundle over  $B$ ) as the pull-back  $s^*A$ , and this implies that one has a generalized geodesic flow  $\gamma_\tau$  on  $T^t G$ . Now for  $X \in \mathfrak{g}$ ,

$$e^X = pr_t(\gamma_1(X)), \quad (5.5)$$

where on the right-hand side we regard  $X \in \mathfrak{g} \subset T^tG$  via the natural embedding of  $\mathfrak{g} \equiv \ker(t_*)[T_{G_0}G$  in  $T^tG$ . If  $G$  is a Lie group then obviously no connection needs to be chosen (all vectors on  $\mathfrak{g}$  are vertical, so the geodesic flow on  $\mathfrak{g}$  is the identity map), and the map  $\exp$  reduces to the usual one.

### 5.2. Algebras of observables from Lie algebroids and groupoids

Generalizing the Poisson algebra  $C^\infty(\mathfrak{g}^*)$  of a Lie algebra  $\mathfrak{g}$  (which is a vector bundle over a single point), one may associate a Poisson algebra  $C^\infty(E^*)$  to any Lie algebroid  $E$  [16]; here  $E^*$  is the dual of  $E$  as a vector bundle. The Poisson structure is completely determined by specifying the Poisson bracket between arbitrary sections  $\xi_1, \xi_2$  of  $E$  and functions  $f_1, f_2$  on  $B$ . Here any  $\xi \in \Gamma(E)$  defines an element  $\tilde{\xi} \in C^\infty(E^*)$  as follows: if  $pr$  is the projection in  $E^*$  then  $\tilde{\xi}(\theta) = \langle \theta, \xi(pr(\theta)) \rangle$ . These functions  $\tilde{\xi}$  are obviously linear on the fibers of  $E^*$ . Furthermore,  $f \in C^\infty(B)$  defined  $\tilde{f} \in C^\infty(E^*)$  by pull-back. The Poisson brackets are

$$\{\tilde{\xi}_1, \tilde{\xi}_2\} = [\widetilde{\xi_1, \xi_2}]; \quad \{\tilde{f}_1, \tilde{f}_2\} = 0; \quad \{\tilde{\xi}, \tilde{f}\} = q(\widetilde{\xi})f. \quad (5.6)$$

This bracket may subsequently be extended to a dense subset of  $C^\infty(E^*)$  (in a suitable topology) by imposing the Leibniz rule on products of linear functions. On  $E = TQ$  this procedure is equivalent to imposing the identities  $\{\sigma(\xi_1), \sigma(\xi_2)\} = \sigma([\xi_1, \xi_2])$ ,  $\{\tilde{f}_1, \tilde{f}_2\} = 0$ , and  $\{\sigma\xi, \tilde{f}\} = \xi(f)$ , where  $\sigma(\xi) \in C^\infty(T^*Q)$  is the symbol of the vector field  $\xi$  on  $Q$ . This leads to the canonical Poisson structure on  $C^\infty(T^*Q)$ . In case that  $E = \mathfrak{g}$  is the Lie algebroid of a Lie groupoid, a more intrinsic construction of this Poisson structure is given in [16, II.4.2].

In similar spirit, we can construct a non-commutative  $C^*$ -algebra (hence a non-associative Jordan-Lie algebra) from a Lie groupoid  $G$  (indeed, from almost any topological groupoid [37], but the construction is more canonical in the Lie case, where a natural measure class is singled out, see below). To do so, we need to choose a measure  $\mu_b$  on each fiber  $t^{-1}(b)$  of  $G$ , in such a way that the family of measures thus obtained is left-invariant (that is, the map  $L_x : t^{-1}(s(x)) \rightarrow t^{-1}(t(x))$  should be measure-preserving for all  $x$ ). Since the fibers are manifolds, we naturally require that each measure  $\mu_b$  is equivalent to the Lebesgue measure (on a local chart). The precise choice of the  $\mu_b$  does not matter very much in that case, as the  $C^*$ -algebras corresponding to different such choices will be isomorphic. In both the Lie and the general case, groupoid  $C^*$ -algebras are of major mathematical interest, as they provide fascinating examples of non-commutative geometry (cyclic cohomology) and topology (K-homology), cf. [13, 32]. The algebra is constructed starting from  $C_c^\infty(G)$ , which is equipped with a product

$$f * g(x) = \int_{t^{-1}(s(x))} d\mu_{s(x)}(y) f(xy)g(y^{-1}), \quad (5.7)$$

and an involution

$$f^*(x) = \overline{f(x^{-1})}, \quad (5.8)$$

which are clearly generalizations of (3.19). The construction of the norm is described in [37] (for general groupoids), and the closure of  $C_c^\infty(G)$  in this norm is the groupoid algebra  $C^*(G)$ . Its self-adjoint part, with the multiplications  $\sigma_\hbar$  and  $\alpha_\hbar$  (cf. (3.5)), is the Jordan-Lie algebra  $JL(G)$ .

For  $G$  a Lie group we thus recover the group algebra, whose representation theory is discussed in subsect. 3.4; the opposite case is the so-called coarse groupoid  $G = Q \times Q$ , where  $Q$  is a manifold. This has base space  $B = Q$ , and source and target projections  $s((x, y)) = y$ ,  $t((x, y)) = x$ . The inclusion is  $i(x) = (x, x)$ , the inverse is  $(x, y)^{-1} = (y, x)$ , and the composition rule is  $(x_1, y)(y, x_2) = (x_1, x_2)$ . The measures  $\mu_b$  may all be taken to be identical to a single measure  $\mu$  on  $Q$ , and one easily finds that  $C^*(Q \times Q) = \mathcal{K}(L^2(Q; \mu))$ , cf. [13]. Its self-adjoint subspace  $JL(Q \times Q)$  is the quantum algebra of observables of a particle moving on  $Q$  [27], and it will not come as a surprise that the Poisson algebra of the Lie algebroid  $TQ$  of  $Q \times Q$  is just  $C^\infty(T^*Q)$ , the classical algebra of observables of the particle. The quantum algebra  $JL(Q \times Q)$  has only one irreducible representation, namely the defining one on  $L^2(Q; \mu)$  (up to unitary equivalence). Similarly, the classical algebra  $C^\infty(T^*Q)$  has only one classical irreducible representation (up to symplectomorphisms), given by  $S = T^*Q$ . These are the quantum as well as classical Jordan-Lie analogues of the well-known Stone- von Neumann uniqueness theorem on regular representations of the canonical commutation relations (see e.g. [10, 18, 15] for this theorem in its various settings).

The situation where  $G$  is either a Lie group, or the coarse groupoid of some manifold, are both special cases of so-called gauge groupoids [31, 16]. A gauge groupoid is equivalent to a principal fibre bundle  $(P, Q, H)$ , where  $P$  is the total space,  $Q$  is the base space, and  $H$  is a Lie group acting on  $P$  from the right. The corresponding groupoid is denoted by  $P \times_H P$ . It is a quotient of the coarse groupoid  $P \times P$ , obtained by imposing the equivalence relation  $(x_1, x_2) \sim (y_1, y_2)$  iff  $(x_1, x_2) = (y_1 h, y_2 h)$  for some  $h \in H$ ; we denote the equivalence class of  $(x, y)$  by  $[x, y]$ . Accordingly,  $B = Q = P/H$ , the inverse is  $[x, y]^{-1} = [y, x]$ , the projections are  $s([x, y]) = pr_{P \rightarrow Q}(y)$ ,  $t([x, y]) = pr_{P \rightarrow Q}(x)$ , the inclusion is  $i(q) = [s(q), s(q)]$  (for an arbitrary section  $s$  of  $P$ ), and multiplication  $[x_1, y_1] \cdot [y_2, x_2]$  is defined iff  $y_2 = y_1 h$  for some  $h \in H$ , and the composition equals  $[x_1 h, x_2]$  in that case. For  $H = \{e\}$  we get the coarse groupoid, and for  $P = H = G$  we get a Lie group  $G$ . It can be shown that any transitive groupoid is of the form  $P \times_H P$  [31].

If  $H$  is compact the groupoid  $C^*$ -algebra is  $C^*(P \times_H P) \simeq C^*(Q \times Q) \otimes C^*(H)$  [27], which is the quantum algebra of observables of a particle moving on  $Q$  with an internal degree of freedom, namely a charge coupling to a gauge field defined on the bundle  $(P, Q, H)$ . The Lie algebroid of  $P \times_H P$  is  $(TP)/H$  (where the action of  $H$  on  $TP$  is the push-forward of its action on  $P$ ). The corresponding Poisson algebra  $C^\infty((T^*P)/H)$  was already known to be the classical algebra of observables of a particle coupling to a Yang-Mills field [20],

and it is satisfying that the quantum algebra  $C^*(P \times_H P)_{\text{sa}}$  can be obtained as a deformation of it; the quantization maps  $Q_{\hbar}$  are given in [27].

The irreducible representations of the classical algebra of observables  $A_0 = C^\infty((T^*P)/H)$  correspond to the symplectic leaves of  $(T^*P)/H$  (and their covering spaces), which are discussed in [20]. There is a one-to-one correspondence between the set of these leaves, and the set of co-adjoint orbits in  $\mathfrak{h}^*$  (the dual of the Lie algebra of  $H$ ): each leaf  $P_{\mathcal{O}}$  is a fiber bundle over  $T^*Q$  whose characteristic fiber is the co-adjoint orbit  $\mathcal{O}$ . Hence locally  $P_{\mathcal{O}} \simeq T^*Q \times \mathcal{O}$ , and the orbit  $\mathcal{O} \subset \mathfrak{h}^*$  clearly serves as a classical internal degree of freedom ('charge') of the particle. Hence the representation theory of  $C^\infty((T^*P)/H)$  is isomorphic to that of  $C^\infty(\mathfrak{h}^*)$  with the Lie etc. Poisson structure discussed in subsect. 2.3.

An analogous situation prevails in the quantum case  $\mathfrak{A} = \mathfrak{JL}(\mathfrak{P} \times_{\mathfrak{H}} \mathfrak{P})$  [27]. The representation theory of this algebra is isomorphic to that of  $JL(H)$  (see subsect. 3.4), hence each irreducible unitary representation  $\pi_\chi$  of  $H$  on a Hilbert space  $\mathcal{H}_\chi$  induces an irreducible representation  $\pi^\chi$  of  $\mathfrak{A}$ , and *vice versa*. The Hilbert space  $\mathcal{H}^\chi$  carrying the representation  $\pi^\chi(\mathfrak{A})$  is naturally realized as  $\mathcal{H}^\chi \simeq L^2(Q) \otimes \mathcal{H}_\chi$ , so that we see that  $\mathcal{H}_\chi$  acts as an internal degree of freedom of the particle (a 'quantum charge').

To sum up, we see that classical internal degrees of freedom are co-adjoint orbits of a Lie group, whereas the quantum analogue of this is an irreducible unitary representation of the same group, compare with the discussion in subsect. 3.4.

We end in a speculative manner. In [27] one finds a proof of the transitive case of the following

**CONJECTURE 1.** *Let  $G$  be a Lie groupoid, and  $\mathfrak{g}$  its Lie algebroid. Then there exists a quantization relating the Poisson algebra  $C^\infty(\mathfrak{g}^*)$  canonically associated to  $\mathfrak{g}$  to the Jordan-Lie algebra  $JL(G) = C^*(G)_{\text{sa}}$ , in the sense of Definition 7.*

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